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HENRY BYRON NEWSON, Ph. D.

THEORY OF
COLLINEATIONS.

By
HENRY BYRON NEWSON, Ph. D.,
Late Professor of Mathematics
in the University of Kansas.

TOPEKA, KANSAS.
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1911.

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PREFACE.

In laying before the public the present work on the Theory of Collineations, I wish to say a word on the historical development of the subject and the genesis of my own interest in it, as well as a word on the point of view I have adopted and the methods I have used.

The concept and term collineation* were introduced into geometry by Möbius in his *Barycentrische Calcul* published at Leipzig in 1827. According to his definition of a collineation points correspond to points and straight lines to straight lines, *i. e.*, collinear points to collinear points, whence the name.

We owe to Möbius not only the first clear-cut notion of a collineation and its name, but also the fundamental theorem underlying all his work on this subject, viz., that the cross-ratios of four corresponding elements of two collinear figures are always equal. He also gives us methods for constructing collineations on a line, in a plane, and in space. He shows that three points on a line, four points in a plane, five points in ordinary space, in general $n+2$ points in a space of n dimensions, determine a collineation in these spaces, respectively. He points out that two conics in a plane are always collinear to one another in ∞^3 ways; and that a curve of the nth degree corresponds to a curve of the same degree. But I find no hint anywhere in Möbius's work that there are any self-corresponding points, lines, or planes in a collineation.

With the introduction of homogeneous coordinates into analytic geometry there came in a generalized form the old problems connected with the transformation of coordinate axes. Such a transformation is a linear transformation, and hence the theory of linear transformations came to be studied

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*Möbius tells us in his *Vorrede*, p. xii, that the name was suggested to him by his friend, Professor Weiske.

as a subsection of modern analytic geometry. A forward step in the theory of collineations was taken by the English school of mathematicians who founded the invariant theory of linear transformations. This theory took its rise shortly after 1840, and the principal names associated with its early development are those of Boole, Cayley, Sylvester, Salmon.*

Since a linear transformation is a projective transformation, every theorem concerning linear transformations has its bearing on the theory of collineations. The workers in projective invariant theory who considered the geometric applications of their science, looked more to the effect of a linear transformation on a geometric figure than to the properties of the transformation itself. Thus we look in vain through the standard works on invariant theory for a classification of linear transformations or a discussion of their characteristic properties. It was left to men with a different point of view to call the attention of the mathematical world from the effects of a collineation back to the properties of the collineation itself.

In 1844 Hermann Grassmann published his *Ausdehnungslehre*, or Calculus of Extension, and a second presentation of the same subject in 1862. The method of the Calculus of Extension was not applied directly by Grassmann to the study of collineations, but it is capable of application to some phases of the subject. For example, by this method the various types of collineations in ordinary space have been determined. Although the contributions of Grassmann's theory to the theory of collineations have been relatively small, they are perhaps sufficient to warrant the mention of it among the analytic methods of treating the subject of collineations.

The quaternion calculus invented by Sir William R. Hamilton, and published by him in his Lectures on Quaternions in 1844, is an algebra founded on a complex number system of four units. One of its valuable applications is to the theory of homogeneous strains. A homogeneous strain is by definition a collineation, though of a very special kind, viz., one which leaves the plane at infinity invariant. However, the quaternion calculus has not been extensively applied to the theory of collineations in ordinary space, probably because it has not been found to be a suitable instrument for the purpose.

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* See note to Salmon's Algebra, chapter XIII.

We mention next an analytic method whose most natural and obvious geometrical application is to the theory of collineations. I refer to Cayley's theory of matrices. This theory was set forth in his memoir on this subject in 1858. This subject has never become a popular one among mathematicians in the sense that it has attracted a large number of independent investigators. It did not lead its founder to the general theory of collineation groups, although it has contributed largely, through the labors of Frobenius and others, to some phases of group theory.

In his *Geometrie der Lage*, Nuremberg, 1847–'60, Von Staudt laid the foundations of pure projective geometry in a form independent of the assumptions of measurement, mechanics or congruence, and without quantitative notions of any sort. He distinguishes sharply between Geometrie der Lage and Geometrie des Masses. Pure projective geometry and the theory of collineations may be considered in a certain sense as mutually inclusive sciences. My conception of the distinction between them is expressed by saying that projective geometry deals chiefly with the projective properties of figures, while the theory of collineations considers especially the properties of the projection itself.

About the year 1870 there appeared upon the mathematical stage a new personality, Sophus Lie, from the land of Abel. He brought with him a new and original idea, the notion of a continuous group of transformations. Lie broadened and deepened the already existing notions of a transformation, and developed a complete theory of all continuous groups of transformations, a thirty years' task. Among the many transformations studied by Lie, the first, the simplest, the most centrally situated, and the most far-reaching in its theoretical and practical bearings, are projective transformations or collineations.

Lie's work on the theory of collineations was both synthetical and analytical; synthetical in its earliest conception and announcement, analytical in its final form as presented to the mathematical world in the books published in his later years. Lie throughout kept his eye fixed on the properties of the collineation itself rather than on the effect of the collineation on certain configurations of space But it is evident that his chief interest in projective transformations was in their group properties, and not in those more fundamental properties which form the natural basis for a classification both of collineations and their groups.

But after all is said the most important and most interesting properties of collineations are their group properties; and no discussion of the theory of collineations is full and symmetrical which fails to lay the major stress on the consideration of the collineation groups. The group of projective transformations, or collineations, is by far the most important of the continuous groups discovered by Lie and developed by him in his "*Theorie der Transformationsgruppen.*" This group lies at the very heart and core of his theory for the reason that all finite continuous groups can, by a suitable transformation of variables, be shown to be similar in structure to some projective group. Therefore every contribution to our knowledge of collineations and their groups reacts upon the wider theory of all continuous groups. A transformation of the elements of a space is defined as an operation which interchanges among themselves the elements of a space, but leaves the space, considered as the aggregate of all its elements, unchanged as a whole. The operation may be produced by means of a mechanical device, an analytical formula, a geometrical construction, or in any other way. Sometimes there are several different methods of producing one and the same transformation; but the effect is the same no matter by what method produced. A collineation is defined as one that transforms points into points, lines into lines, and planes into planes. It is, therefore, a self-dualistic transformation.

A collineation may be regarded from two distinct points of view, viz., the analytic and the synthetic. From the synthetic point of view the phenomena of a collineation appeal directly to the eye or to the space intuitions. On the other hand, from the analytic point of view the operation is seen through the medium of a linear substitution on the requisite number of variables. The two methods have long been in use side by side and each has its special advantages. Each also has its special votaries, and each will continue to have its advocates as long as human minds continue to be constructed on different patterns. To me the synthetic method is the more attractive, for the reason that it enables one to get closer to the facts and to view them at first hand. In all applications of analysis to geometry a formula is only the vehicle which conveys the thought, not the thought itself. The inevitable tendency is to confuse the vehicle with the thought, to mistake the vessel for the contents, and to lay hold on the shadow rather than the substance of the thing sought.

My interest in the collineation as an object of research dates from the time when it was my rare good fortune to be a student of Lie at Leipzig in 1887–'88. I followed with special interest his lectures on Modern Geometry and on Continuous Groups. The latter course was afterward published under the title *Vorlesungen ueber Continuerliche Gruppen.* Almost every example used to illustrate the theory of continuous groups was a group of projective transformations. Lie's method of approach to the theory of projective groups was through the infinitesimal transformation. I early became dissatisfied with the infinitesimal method because there seemed to me so wide a gap between the analytic processes and the geometric interpretation of the results. I was constantly asking myself the question, whether it was not possible to develop the theory of the projective group directly from the finite form of the equations of a linear transformation or from geometric construction? Lie's analytic method started from the finite form of a linear transformation, descended into the infinitesimal regions where the important analytic work was done, then reascended into the regions of the finite where the results were exhibited.

H. B. NEWSON.

NOTE.

The above incomplete draft of the preface probably includes nearly all, except acknowledgments, that the author intended to say. Otherwise the manuscript of this volume was complete and the proof had been read and corrected through to page 272, when his sudden death on the night of February 17, 1910, put an end to his labors. Others, have read the remainder of the proof. Doubtless, errors have crept in which the author would have corrected if he had lived to read the proof himself. It is to be regretted that a series of unfortunate circumstances has so long delayed the publication of this work.

Thanks are due to Dr. Paul Wernicke for assistance rendered the author both with the manuscript and with the proof-reading, also for reading a considerable portion of the remaining proof. I also wish to express my thanks to Dr. U. G. Mitchell for valuable assistance, without which the completion of the publication might not have been possible.

M. W. NEWSON.

CHAPTER I.

PROJECTIVE TRANSFORMATIONS IN ONE-DIMENSIONAL SPACE.

§ 1. General Properties of One-Dimensional Projective Transformations.
§ 2. Types and Normal Forms of Projective Transformations.
§ 3 One-Parameter Groups of Projective Transformations.
§ 4. Two- and Three-Parameter Groups of Projective Transformations.
§ 5. Transformations of Pencils of Lines and Planes.
§ 6. Real Projective Transformations.
§ 7. Theory of Projection.
§ 8. Geometric Theory of Projective Transformations.
§ 9. Geometric Theory of Transformations of Pencils of Lines and Planes.
Exercises.

1. The present chapter is devoted to an exposition of the theory of projective transformations in space of one dimension. The theory applies equally well to all three one-dimensional primary forms of projective geometry, viz., a range of points on a line, a pencil of lines through a point, and a pencil of planes through a line. The principal facts of one-dimensional projective transformations are set forth and on these are built a comprehensive theory of their continuous groups. The theory in one-dimension is sufficiently complete to serve as a foundation and model on which to build a consistent theory of collineations in two, three and higher dimensions.

In §1 we shall define analytically a projective transformation in one dimension. In §§1 to 5 are developed the consequences of this analytic definition, and in §6 is considered the special case when the variables and coefficients in the equation are all real quantities. A geometrical theory of one-dimensional projective transformations is developed in §§7 to 9, and the two theories, analytic and geometric, are shown to be in perfect harmony. Each method will be seen to have its special points of advantage. The chapter closes with a classified list of exercises illustrating both methods.

§1. General Properties of One-Dimensional Projective Transformations.

2. *Analytic Definition.* A projective transformation in one dimension is defined analytically by the equation

$$T : x_1 = \frac{ax+b}{cx+d}; \tag{1}$$

or, in homogeneous coordinates by the pair of equations

$$\begin{aligned} \rho x_1 &= ax + by, \\ \rho y_1 &= cx + dy. \end{aligned} \tag{1'}$$

In these equations the coefficients, a, b, c, d, are constants, and the variables are x, x_1, and x, y, x_1, y_1. Both constants and variables are to be regarded as complex numbers, unless otherwise expressly stated.

The determinant $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ is called the determinant of the transformation T; it is assumed, for the present at least, that this determinant does not vanish.

A projective transformation of the points on a line should be looked upon as an operation which, when applied to a finite set of points or to the range of all points on the line, has the effect of rearranging and redistributing the points of the set or range so as to form a new set or a new range. The sets or ranges of points which are related to one another by a projective transformation are said to be projective.

3. *A Transformation and its Inverse.* The transformation T expressed by equation (1) transforms the point x into x_1, where x is any point on the line. Equation (1) may be solved for x, giving us

$$x = \frac{-dx_1 + b}{cx_1 - a}. \tag{2}$$

The transformation expressed by this equation is called the inverse of T and is symbolized by T^{-1}. T^{-1} transforms a point x_1 into x. The two transformations T and T^{-1} are so related to each other that if T transforms a point P into P_1, T^{-1} transforms P_1 back to P.

4. *Invariant Points.* When the points of a line are shifted into new positions by a projective transformation T, does it ever happen that one or more of the points are unaltered in position? To answer this question, we reason as follows: The coordinate of a point x, which remains at rest or unaltered in position, *i. e.*, which is transformed into itself, must satisfy the equation

$$x = \frac{ax+b}{cx+d}.$$

Clearing of fraction, we see that the coordinates of all such points satisfy the quadratic equation

$$cx^2 + (d-a)x - b = 0; \qquad (3)$$

whence we conclude that a projective transformation T leaves unaltered two points on the line, and their coordinates are given by the roots of equation (3). These two points are generally distinct, but for special values of a, b, c, d, they may coincide. They are called the *invariant points* of the transformation. Two transformations will not generally have the same invariant points; but, as we shall learn, an unlimited number of transformations may have one or both invariant points in common.

There is one particular transformation that leaves every point of the line invariant. If $b=c=0$ and $d=a$ in equation (1), we get $x_1 = x$. This shows that x always equals x_1 or that every point on the line is transformed into itself. This transformation is called the identical projective transformation.

THEOREM 1. **A projective transformation of the points on a line leaves invariant either two distinct points, two coincident points, or all points on the line.**

5. *Characteristic Equation of T.* Let T be given in the homogeneous form as follows:

$$\begin{aligned} \rho x_1 &= ax + by, \\ \rho y_1 &= cx + dy. \end{aligned} \qquad (1')$$

We indicate another way of finding the invariant points of T. Set $x_1 = x$ and $y_1 = y$ in the above equations and transpose; thus we get

$$\begin{aligned}(a-\rho)x+by&=0,\\ cx+(d-\rho)y&=0.\end{aligned}\qquad(4)$$

If these equations are simultaneous, their resultant vanishes; thus

$$\begin{vmatrix}a-\rho & b\\ c & d-\rho\end{vmatrix}=0.$$

Developing the determinant we get the quadratic equation,

$$\rho^2-(a+d)\rho+(ad-bc)=0,\qquad(5)$$

which is called the characteristic equation of T; its roots may be equal or unequal.

In the first case, suppose that equation (5) has two distinct roots, ρ_1 and ρ_2. If one of these roots as ρ_1 be substituted for ρ in (4), these become simultaneous and may be solved for the ratio $x:y$. The value of this ratio $x:y$ gives the coordinates of the invariant point corresponding to ρ_1. If ρ_2, the other root of the characteristic equation, be substituted for ρ in (4), these again become simultaneous and their common solution gives the coordinates of the invariant point corresponding to ρ_2.

In the second case, suppose the characteristic equation (5) has a pair of equal roots. Then there is only one value of ρ which, when substituted in equation (4), makes them simultaneous. It follows in this case that T has only one invariant point, or as we may say, two coincident invariant points.

6. *Pseudo-transformations.*—If the determinant of T vanishes, the transformation is called a *pseudo-transformation.* In defining the transformation it was expressly stated that the determinant must not be zero. This condition excludes just these transformations called pseudo-transformations. The equation of the transformation is written

$$x_1=\frac{ax+b}{cx+d};$$

if the determinant $ad-bc=0$, then $d=\frac{bc}{a}$. Substituting this value of d in the equation, we have

$$x_1 = \frac{a(ax+b)}{c(ax+b)} = \frac{a}{c}; \tag{6}$$

which shows that every point on the line is transformed into the fixed point $\frac{a}{c}$. The inverse of the transformation T is written

$$x = \frac{-dx_1+b}{cx_1-a}.$$

The determinant of this is also $ad-bc$, which equated to zero also gives $d=\frac{bc}{a}$. Substituting this value of d in the last equation, we have

$$x = -\frac{b(cx_1-a)}{a(cx_1-a)} = -\frac{b}{a}; \tag{7}$$

which shows that every point on the line is transformed by the psuedo-transformation (7) into the fixed point $-\frac{b}{a}$.

The invariant points of a pseudo-transformation are also given by equation (3). Putting $d=\frac{bc}{a}$ in this equation, it breaks up into

$$\left(x-\frac{a}{c}\right)\left(x+\frac{b}{a}\right)=0; \tag{8}$$

thus showing that $\frac{a}{c}$ and $-\frac{b}{a}$ are the invariant points of the pseudo-transformation.

THEOREM 2. A pseudo-transformation transforms every point on the line into one or the other of its invariant points.

7. *Three Conditions Determine a Projective Transformation.* The equation of a projective transformation T contains three independent constants, viz., $a:b:c:d$. We infer, therefore, that three conditions determine such a transformation. In particular, three points and their corresponding points determine uniquely and completely a projective transformation.

Let x', x'', x''' be any three points on a line, and x_1', x_1'', x_1''' their corresponding points, respectively. Substituting successively in (1) the coordinates of each pair of corresponding points, we have three equations, viz.:

$$\begin{aligned} cx'x_1' &+ dx_1' - ax' - b = 0, \\ cx''x_1'' &+ dx_1'' - ax'' - b = 0, \\ cx'''x_1''' &+ dx_1''' - ax''' - b = 0. \end{aligned} \tag{9}$$

These equations are linear and homogeneous in a, b, c, d, and determine the ratios of these quantities uniquely and completely, provided no two of these equations are identical or have their coefficients proportional.

THEOREM 3. There is one and only one projective transformation that transforms three given points on a line into three other given points.

8. *The Identical Transformation.*—Suppose that the transformation (1) leaves three points of the line invariant. If we put $x_1'=x'$, $x_1''=x''$ and $x_1'''=x'''$ in equations (9), these reduce to the following:

$$\begin{aligned} cx'^2 &+ (d-a)x' - b = 0, \\ cx''^2 &+ (d-a)x'' - b = 0, \\ cx'''^2 &+ (d-a)x''' - b = 0. \end{aligned} \tag{10}$$

The determinant of these equations,

$$\begin{vmatrix} x'^2 & x' & 1 \\ x''^2 & x'' & 1 \\ x'''^2 & x''' & 1 \end{vmatrix} \equiv (x'-x''')\,(x''-x''')\,(x'-x''), \tag{11}$$

does not vanish so long as the three points are distinct; consequently, the coefficients of the above equations must vanish identically. Thus, $c=0$, $b=0$, $d=a$. Putting these values in (1) we get $x_1=x$, which is the identical transformation. The identical transformation we know transforms every point of the line into itself.

THEOREM 4. A projective transformation which leaves three points of a line invariant is the identical transformation and leaves all points of the line invariant.

9. *Invariance of Cross-ratio.* Let x, x', x'', x''', be the coordinates of any four points on the line. The function $\frac{x''-x}{x''-x'} : \frac{x'''-x}{x'''-x'}$, is called the *cross-ratio* (*Doppelverhaltniss, ratio anharmonique*) of the four points. Let k be the value of

this function; we shall designate the cross-ratio by the symbol $k=(x\,x'\,x''\,x''')$.

Let the four points x, x', x'', x''', be transformed into x_1, x_1', x_1'', x_1''' respectively by the projective transformation

$$x_1=\frac{ax+b}{cx+d}.$$

We wish to compare the cross-ratio of these four points with that of their four corresponding points. To do this we have only to substitute (1) in the cross-ratio function and reduce the resulting expression. Thus

$$k=\frac{x_1''-x_1}{x_1''-x_1'}:\frac{x_1'''-x_1}{x_1''-x_1'}=\frac{\dfrac{ax''+b}{cx''+d}-\dfrac{ax+b}{cx+d}}{\dfrac{ax''+b}{cx''+d}-\dfrac{ax'+b}{cx'+d}}:\frac{\dfrac{ax'''+b}{cx'''+d}-\dfrac{ax+b}{cx+d}}{\dfrac{ax'''+b}{cx'''+d}-\dfrac{ax'+b}{cx'+d}}$$

$$=\frac{(ad-bc)\,(cx'+d)\,(x''-x)}{(ad-bc)\,(cx+d)\,(x''-x')}:\frac{(ad-bc)\,(cx'+d)\,(x'''-x)}{(ad-bc)\,(cx+d)\,(x'''-x')}$$

$$=\frac{x''-x}{x''-x'}:\frac{x'''-x}{x'''-x'}.$$

Hence we see that the cross-ratio of four points on a line is unaltered by a projective transformation of the points on the line.

THEOREM 5. A projective transformation of the points on a line leaves invariant the cross-ratio of any four points on the line.

10. *Resultant of Two Transformations.*—Let T and T_1 be two transformations whose equations are respectively

$$x_1=\frac{ax+b}{cx+d}\text{ and }x_2=\frac{a_1x_1+b_1}{c_1x_1+d_1}. \tag{1}$$

The first transforms the point x into x_1, and the second transforms x_1 into x_2. We suppose the operations are carried out in the order in which the equations are written. If we eliminate x_1 from the above, we get

$$x_2=\frac{(a_1a+b_1c)\,x+(a_1b+b_1d)}{(c_1a+d_1c)\,x+(c_1b+d_1d)}. \tag{12}$$

It should be observed that (12) is of the same form as (1) and differs from it only in the values of the coefficients. Equation (12) therefore expresses a projective transformation T_2, which transforms the point x directly to x_2, and is equivalent to the successive applications of T and T_1 in the order named. The transformation T_2 is called the *resultant* of the transformations T and T_1, which are called the component transformations. The operation is symbolized thus: $TT_1=T_2$. If the two component transformations T and T_1 are taken in the reverse order, the resultant, $T_1T=T_2'$, is not the same as T_2. Thus:

$$T_2' : x_2 = \frac{(aa_1+bc_1)x+(ab_1+bd_1)}{(ca_1+dc_1)x+(cb_1+dd_1)}, \tag{12'}$$

which is not the same as T_2. The two projective transformations T_2 and T_2' are called *conjugate* transformations.

By referring to the transformations lettered T, T_1, T_2, we see that the determinant of T_2 is

$$\begin{vmatrix} a_1a+b_1c & a_1b+b_1d \\ c_1a+d_1c & c_1b+d_1d \end{vmatrix};$$

but this is the product of

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} \text{ by } \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix};$$

these determinants are respectively the determinants of T and T_1, the components of T_2. Hence the determinant of a transformation, T_2, which is the resultant of transformations T and T_1, is equal to the product of the determinants of T and T_1.

This result is capable of immediate extension; for let T_a, T_b and T_c denote three transformations, the result of whose successive applications is equivalent to T_d; the compounding of T_a and T_b is equivalent to a third transformation, T_{ab}. The resultant of T_{ab} and T_c is T_d, and the determinant of T_d is equal to the product of the determinants of T_{ab} and T_c; hence the determinant of T_d is equal to the product of those of T_a, T_b, and T_c. This mode of reasoning is applicable to the resultant of any number of transformations; hence by induction we infer the following theorem:

THEOREM 6. The resultant T_n of n projective transformations T_i $(i=0, 1, 2, \ldots . n-1)$ is a projective transformation, and the determinant of the resultant is equal to the product of the determinants of the components.

11. *Commutative Transformations.* Two transformations T and T_1 are said to be *commutative* when the resultants, taken in either order, are equal; *i. e.*, when T_2 and T_2' of the last article give the same transformation. We may find the conditions that must be satisfied in order that T and T_1 are commutative by equating corresponding coefficients in equations (12) and (12′). We thus get

$$\frac{a-d}{a_1-d_1}=\frac{b}{b_1}=\frac{c}{c_1},$$

as the necessary conditions of commutativity.

The invariant points of T and T_1 are given by the roots of the respective equations, art. 4,

$$cx^2-(a-d)x-b=0 \text{ and } c_1x^2-(a_1-d_1)x-b_1=0\,.$$

The conditions of commutativity show that these two equations have the same roots. It will be shown later that these necessary conditions are also sufficient. Hence

THEOREM 7. Two projective transformations T and T_1 are commutative when and only when they have the same invariant points.

§2. Types and Normal Forms of Projective Transformations.

12. *Two Types of Projective Transformations.* The invariant points of a transformation T are given by the roots of the quadratic equation (3). The roots of this equation are:

$$(A, A')=\frac{a-d\pm\sqrt{(a+d)^2-4(ad-bc)}}{2c} \tag{13}$$

These two roots are distinct or coincident, according as

$$(a+d)^2-4(ad-bc)\neq 0,$$

or

$$= 0.$$

Thus there are two distinct types of transformation. The first type is characterized by the fact that it has two invariant points, while the second type has only one. Every transformation, not identical, belongs to one or the other of these types.

13. *Implicit Normal Form of Type I.* A transformation T of type I, whose invariant points are A and A', may be written in the form:

$$\frac{x_1-A}{x_1-A'}=k\frac{x-A}{x-A'}, \tag{14}$$

where the constant k is expressible in the terms of the coefficients, a, b, c, d, as follows:

$$k=\frac{\left(a+d-\sqrt{(a+d)^2-4(ad-bc)}\right)^2}{4(ad-bc)}. \tag{15}$$

To show this, solve equation (14) for x_1; this gives us

$$x_1=\frac{(A-kA')x-AA'(1-k)}{(1-k)x-(A'-kA)}, \tag{16}$$

which is of the same form as

$$x_1=\frac{ax+b}{cx+d}.$$

Comparing the coefficients of these forms, we have

$$\frac{A-kA'}{1-k}=\frac{a}{c},\quad AA'=-\frac{b}{c},\quad \frac{A'-kA}{1-k}=-\frac{d}{c};$$

solving for A, A' and k, we find

$$A=\frac{a-d+\sqrt{(a+d)^2-4(ad-bc)}}{2c},$$

$$A'=\frac{a-d-\sqrt{(a+d)^2-4(ad-bc)}}{2c},$$

$$k=\frac{\left(a+d-\sqrt{(a+d)^2-4(ad-bc)}\right)^2}{4(ad-bc)}, \tag{17}$$

or

$$=\frac{a+d-\sqrt{(a+d)^2-4(ad-bc)}}{a+d+\sqrt{(a+d)^2-4(ad-bc)}},$$

or

$$\frac{(1+k)^2}{k}=\frac{(a+d)^2}{ad-bc}.$$

The values of A and A' thus obtained are the same as the roots of equation (3). Equation (14) is called the *implicit* normal form of type I.

14. *Implicit Normal Form of Type II.*—A transformation of type II, whose single invariant point is A, is reducible to the form

$$\frac{1}{x_1 - A} = \frac{1}{x - A} + t. \tag{18}$$

To verify this, solve for x_1; thus,

$$x_1 = \frac{(1+tA)\,x - tA^2}{tx + (1-tA)}. \tag{19}$$

This is the same form as (1). A and t are found in terms of a, b, c, d, as before, by comparing coefficients and solving for A and t; thus,

$$A = \frac{a-d}{2c} \text{ and } t = \frac{2c}{a+d}. \tag{20}$$

Equation (18) is called the implicit normal form of type II.

THEOREM 8. Every transformation of the form, $x_1 = \frac{ax+b}{cx+d}$, belongs to one or the other of the implicit normal forms,

$$\frac{x_1 - A}{x_1 - A'} = k\,\frac{x - A}{x - A'} \text{ or } \frac{1}{x_1 - A} = \frac{1}{x - A} + t.$$

15. *Geometrical Interpretation of the Normal Forms.*—The normal form of type I may be written:

$$k = \frac{x - A'}{x - A} : \frac{x_1 - A'}{x_1 - A} = (A'Axx_1)\,; \tag{21}$$

i. e., k is the cross-ratio of the four points A', A, x, x_1, where A' and A are the invariant points, and x and x_1 a pair of corresponding points. Here x and x_1 are any pair of corresponding points, and k is a constant quantity.

In the normal form of type II the expressions $x - A$ and $x_1 - A$ are the distances of a pair of corresponding points from the invariant point. The normal form of type II may be written:

$$\frac{1}{x_1 - A} - \frac{1}{x - A} = t, \tag{18'}$$

which shows that the difference of the reciprocals of the distances of a pair of corresponding points from the invariant point is constant for all pairs of corresponding points. Let x be the point at infinity on the line; t is thus seen to be the reciprocal of the segment Ax_1 where x_1 is the point into which the point at infinity is transformed.

THEOREM 9. In a transformation of type I, k, the cross-ratio of the invariant points and a pair of corresponding points, is constant for all pairs of corresponding points; in a transformation of type II, t, the difference of the reciprocals of the distances of a pair of corresponding points, is constant for all pairs of corresponding points.

16. *The Natural Parameters.* When the transformation is written in the form of equation (1), we see that there are three independent parameters viz., $\frac{a}{d}$, $\frac{b}{d}$, $\frac{c}{d}$, when it is of type I; in the case of a transformation of type II, the relation $(a+d)^2=4(ad-bc)$, is satisfied, and there are but two independent parameters. The coefficients, a, b, c, d, have no simple geometric meanings; but in the normal forms A, A', k and A, t have definite important geometric meanings. The parameters A, A', k and A, t are called the *natural* parameters of the transformation.

17. *Explicit Normal Forms.* Equations (16) and (19) may be put into the forms:

$$x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}}, \text{ and } x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ 1 & 0 & tA+1 \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ 1 & 0 & t \end{vmatrix}}. \tag{22}$$

These are called the explicit normal forms of types I and II, respectively.

18. *Determinants of Normal Forms.* The determinant of the explicit normal form of type I is found as follows:

$$x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}} = \frac{(kA'-A)x + AA'(1-k)}{(k-1)x + (A'-kA)}.$$

$$\Delta = \begin{vmatrix} kA'-A & AA'(1-k) \\ k-1 & A'-kA \end{vmatrix} = k \begin{vmatrix} A & 1 \\ A' & 1 \end{vmatrix}^2. \qquad (23)$$

The determinant of the explicit normal form of type II is

$$\Delta = \begin{vmatrix} 1+tA & -tA^2 \\ t & 1-tA \end{vmatrix} = 1. \qquad (24)$$

19. *Type II as the Limiting Form of Type I.* It is evident that type II is the limiting form of type I when the two invariant points coincide. From equation (14) we see that $k = 1$ when $A = A'$. The fraction $\frac{1-k}{A-A'}$ becomes indeterminate when $A = A'$. Putting for A, A' and k their values from (17), we have:

$$\lim_{A'=A} \frac{1-k}{A-A'} = \frac{2c}{a+d}.$$

But from (20) $\frac{2c}{a+d} = t$; hence, $\lim\limits_{A'=A} \frac{1-k}{A-A'} = t$.

By means of this relation the normal form of type II can be deduced directly from that of type I. Dividing both numerator and denominator of (16) by $A-A'$, we get:

$$x_1 = \frac{\frac{(A-kA')}{(A-A')}x - \frac{AA'(1-k)}{(A-A')}}{\frac{(1-k)}{(A-A')}x - \frac{A'-kA}{(A-A')}}.$$

Putting $A'=A$ and $\lim\limits_{A'=A} \frac{1-k}{A-A'} = t$, this reduces to (19).

In the explicit normal form of type I, (22), subtract the second row from the last in each determinant, divide through by $A'-A$, and pass to the limit. In this way we get the explicit normal form of type II.

20. *Characteristic Equation in Normal Form.*—Let T be given in the normal form

$$x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}} = \frac{(kA'-A)x+AA'(1-k)}{(k-1)x+(A'-kA)}.$$

The characteristic equation then becomes

$$\begin{vmatrix} kA'-A-\rho & AA'(1-k) \\ k-1 & A'\;\; kA-\rho \end{vmatrix} = 0\ .$$

Developing this we get as the characteristic equation of T in the normal form

$$\rho^2 + (1+k)\,\Delta\,\rho + k\,\Delta^2 = 0, \tag{25}$$

where Δ is the determinant $\begin{vmatrix} A & 1 \\ A' & 1 \end{vmatrix}$.

The roots of this equation are evidently $-\Delta$ and $-k\Delta$.

The characteristic equation of T' in the normal form is readily found to be

$$\rho^2 - 2\,\rho + 1 = 0. \tag{25$'$}$$

21. *Resultant of T and T_1 in Normal Form.*—Let us next consider the resultant of two transformations T and T_1, both of type I, given in their explicit normal forms in homogeneous coordinates. Let the equations of T, T_1, and T_2, be as follows:

$$T:\ \ \rho x_1 = \begin{vmatrix} x & y & 0 \\ A & B & A \\ A' & B' & kA' \end{vmatrix}, \qquad \rho y_1 = \begin{vmatrix} x & y & 0 \\ A & B & B \\ A' & B' & kB' \end{vmatrix};$$

$$T_1:\ \rho_1 x_2 = \begin{vmatrix} x_1 & y_1 & 0 \\ A_1 & B_1 & A_1 \\ A_1' & B_1' & k_1A_1' \end{vmatrix}, \qquad \rho_1 y_2 = \begin{vmatrix} x_1 & y_1 & 0 \\ A_1 & B_1 & B_1 \\ A_1' & B_1' & k_1B_1' \end{vmatrix};$$

$$T_2:\ \rho_2 x_2 = \begin{vmatrix} x & y & 0 \\ A_2 & B_2 & A_2 \\ A_2' & B_2' & k_2A_2' \end{vmatrix}, \qquad \rho_2 y_2 = \begin{vmatrix} x & y & 0 \\ A_2 & B_2 & B_2 \\ A_2' & B_2' & k_2B_2' \end{vmatrix}.$$

We get the equations of T_2 by substituting x_1 and y_1 from T in T_1; thus,

$$T_2 : \rho\rho_1 x_2 = \begin{vmatrix} \Delta_x & \Delta_y & 0 \\ A_1 & B_1 & A_1 \\ A_1' & B_1' & k_1A_1' \end{vmatrix}, \quad \rho\rho_1 y_2 = \begin{vmatrix} \Delta_x & \Delta_y & 0 \\ A_1 & B_1 & B_1 \\ A_1' & B_1' & k_1B_1' \end{vmatrix};$$

where Δ_x and Δ_y are the determinants in T. These last equations readily become

$$T_2 : \rho\rho_1 x_2 = \begin{vmatrix} x & y & 0 & 0 & 0 \\ A & B & A & B & 0 \\ A' & B' & kA' & kB' & 0 \\ 0 & 0 & A_1 & B_1 & A_1 \\ 0 & 0 & A_1' & B_1' & k_1A_1' \end{vmatrix}; \quad \rho\rho_1 y_2 = \begin{vmatrix} x & y & 0 & 0 & 0 \\ A & B & A & B & 0 \\ A' & B' & kA' & kB' & 0 \\ 0 & 0 & A_1 & B_1 & B_1 \\ 0 & 0 & A_1' & B_1' & k_1B_1' \end{vmatrix}. \qquad (26)$$

Comparing coefficients of x and y in the two forms of T_2 we get the following equations I to IV:

$$\text{(I)} \quad \begin{vmatrix} B_2 & A_2 \\ B_2' & k_2A_2' \end{vmatrix} = \begin{vmatrix} B & A & B & 0 \\ B' & kA' & kB' & 0 \\ 0 & A_1 & B_1 & A_1 \\ 0 & A_1' & B_1' & k_1A_1' \end{vmatrix};$$

$$\text{(II)} \quad \begin{vmatrix} A_2 & A_2 \\ A_2' & k_2A_2' \end{vmatrix} = \begin{vmatrix} A & A & B & 0 \\ A' & kA' & kB' & 0 \\ 0 & A_1 & B_1 & A_1 \\ 0 & A_1' & B_1' & k_1A_1' \end{vmatrix};$$

$$\text{(III)} \quad \begin{vmatrix} B_2 & B_2 \\ B_2' & k_2B_2' \end{vmatrix} = \begin{vmatrix} B & A & B & 0 \\ B' & kA' & kB' & 0 \\ 0 & A_1 & B_1 & B_1 \\ 0 & A_1' & B_1' & k_1B_1' \end{vmatrix};$$

$$\text{(IV)} \quad \begin{vmatrix} A_2 & B_2 \\ A_2' & k_2B_2' \end{vmatrix} = \begin{vmatrix} A & A & B & 0 \\ A' & kA' & kB' & 0 \\ 0 & A_1 & B_1 & A_1 \\ 0 & A_1' & B_1' & k_1B_1' \end{vmatrix};$$

Since the determinant of T_2 is equal to the product of the determinants of T and T_1, (art. 10), we have $\Delta_2 = \Delta\Delta_1$ or using the values of the determinants (art. 18).

$$(\mathrm{V}) \quad k_2 \begin{vmatrix} A_2 & B_2 \\ A_2' & B_2' \end{vmatrix} = kk_1 \begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \cdot \begin{vmatrix} A_1 & B_1 \\ A_1' & B_1' \end{vmatrix}.$$

The system of equations I—IV are not independent, but a system of three independent equations may be obtained from them by dividing any three of them by the fourth. Equation V is not independent of I—IV and may be deduced from them. These equations enable us to determine the natural parameters of T_2 in terms of those of T and T_1. If T, or T_1, or both T and T_1 are of type II, the same process of elimination enables us to determine T_2.

22. *Resultant of T' and T_1' in Normal Form.* The resultant of two transformations of type II is usually of type I, as may readily be shown. We wish to determine the conditions that must be satisfied in order that the resultant of two transformations of type II shall also be of type II. Let the equations of T', T_1', and T_2' be as follows:

$$x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ 1 & 0 & tA'+1 \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ 1 & 0 & t \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} x_1 & 1 & 0 \\ A_1 & 1 & A_1 \\ 1 & 0 & t_1A_1+1 \end{vmatrix}}{\begin{vmatrix} x_1 & 1 & 0 \\ A_1 & 1 & 1 \\ 1 & 0 & t_1 \end{vmatrix}}, \quad x_2 = \frac{\begin{vmatrix} x & 1 & 0 \\ A_2 & 1 & A_2 \\ 1 & 0 & t_2A_2+1 \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A_2 & 1 & 1 \\ 1 & 0 & t_2 \end{vmatrix}}.$$

The resultant of T' and T_1' may also be written in the form

$$x_2 = \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ A & 1 & A & 1 & 0 \\ 1 & 0 & tA+1 & t & 0 \\ 0 & 0 & A_1 & 1 & A_1 \\ 0 & 0 & 1 & 0 & t_1A_1+1 \end{vmatrix} : \begin{vmatrix} x & 1 & 0 & 0 & 0 \\ A & 1 & A & 1 & 0 \\ 1 & 0 & tA+1 & t & 0 \\ 0 & 0 & A_1 & 1 & 1 \\ 0 & 0 & 1 & 0 & t_1 \end{vmatrix}. \qquad (27)$$

This resultant will be of type II when the condition, $(a+d)^2 - 4(ad-bc) = 0$, (art. 12) is satisfied. The determinant of T_2' is equal to the product of the determinants of T' and T_1' which are each equal to 1, hence $\triangle_2 \equiv \begin{vmatrix} a & b \\ c & d \end{vmatrix} = 1$, and the above condition reduces to $(a+d)^2 = 4$. Applying this condition to equation (27) we get in terms of their natural parameters the condition that the resultant of T' and T_1' shall also be of type II, viz.:

$$tt_1(A-A_1)^2 \{ tt_1(A-A_1)^2 - 4 \} = 0.$$

Hence we must have one of the four following cases: $t=0$, $t_1=0$, $A_1=A$, or $tt_1(A-A_1)^2 = 4$.

These necessary conditions are also sufficient. If $t=0$ or $t_1=0$, then T' or T_1' is the identical transformation. The resultant of any transformation T and the identical transformation is evidently T. If $A_1=A$, the two transformations of type II have the same invariant point. Sufficiency of this condition is shown in (art. 28). Finally let any numerical values be assigned to A_1, A, t_1, and t such that $tt_1(A-A_1)^2 = 4$. For example let $A=4$, $t=1$ and $A_1=2$, $t_1=1$. T' reduces to $x_1 = \frac{5x-16}{x-3}$; T_1' becomes $x_2 = \frac{3x_1-4}{x_1-1}$. T_2' is found to be $x_2 = \frac{11x-36}{4x-13}$, which is of type II; $A_2 = 3$ and $t_2 = -4$.

Considering the identical transformation as not properly of type II, we reach the following result:

THEOREM 10. The necessary and sufficient conditions, that the resultant of two transformations of type II should also be of type II, are (1), that they have the same invariant point; or (2), that $tt_1(A-A_1)^2 = 4$.

23. *Symbolic Notation and Operation.* A very useful and convenient symbolism has been invented for dealing with certain transformations and their combinations. We proceed to explain and illustrate this notation.

A transformation is denoted by a single letter T or S. The inverse of T is denoted by T^{-1}. The resultant of T and S is denoted by TS or ST according to the order in which they operate. The resultant of T and T is denoted by T^2. If T be repeated n times the resultant is denoted by T^n, etc. The resultant of T^m and T^n is T^{m+n}. The resultant of T and its inverse is denoted by $TT^{-1} = T^0 = 1$. Thus unity is a convenient symbol for the identical transformation.

A symbolic equation of the form,

$$TSR = ST^{-1}RQ,$$

means that the resultant of the three transformations on the left is equal to that of the five on the right when taken in the order indicated. We may multiply each side of this equation by say T^{-1} by writing T^{-1} before each side; thus

$$T^{-1}TSR = SR = T^{-1}ST^{-1}RQ.$$

Suppose we have two symbolic equations such as

$$S' = T^{-1}ST \text{ and } S_1' = T^{-1}S_1T;$$

by multiplication we get

$$S'S_1' = T^{-1}STT^{-1}S_1T = T^{-1}SS_1T.$$

These examples sufficiently illustrate the principle.

24. *Operation on S by T.* If TS is not the same transformation as ST let us assume that there exists a transformation S' such that $TS' = ST$. Let us multiply both sides of this equation by T^{-1}, the inverse of T. This gives us $T^{-1}TS' = S' = T^{-1}ST$, since TT^{-1} is the identical transformation and denoted by unity. Hence S' is a projective transformation, since it is the resultant of T^{-1}, S and T, each of which is a projective transformation.

The two transformations S and S' are conjugate transformations, as the following equations show: $S' = (T^{-1}S)T$ and $S = T(T^{-1}S)$, *i. e.*, $(T^{-1}S)$ and T combined in one order give S', and in the reverse order give S.

We say that the transformation S' is obtained by *operating* on S with T. Thus the operation of T on S produces S' and is symbolized by $S' = T^{-1}ST$. This equation may be solved for S, so to speak, by the following process: Write T before each side of $S' = T^{-1}ST$ and we get $TS' = ST$. Now write

T^{-1} after each side of this equation and we get $TS'T^{-1} = S$, which shows the relation between S and S'.

25. *Equations of S and S'.* Let S be given by the equation $x_1 = \frac{a_1 x + b_1}{c_1 x + d_1}$ and let us operate on S by T, given by $x_1 = \frac{ax+b}{cx+d}$. Since S' is given by $S' = T^{-1}ST$, we find the equation for S',

$$x_1 = \frac{(-ada_1 - bdc_1 + acb_1 + bcd_1)\,x + (aba_1 + b^2c_1 - a^2b_1 - abd_1)}{(-cda_1 - d^2c_1 + c^2b + cdd_1)\,x + (bca_1 + bdc_1 - acb_1 - add_1)}$$
$$= \frac{a_1' x + b_1'}{c_1' x + d_1'}. \qquad (28)$$

Let the natural parameters of S be k, A, and A'; and of S', k_1, A_1, and A_1'; we wish to find the relation existing between the natural parameters of S and S'.

From the equations of S' and S we readily find

$$a_1' + d_1' = -\Delta(a_1 + d_1), \qquad (29)$$

where Δ is the determinant of T. Also from theorem 6 we have,

$$\begin{vmatrix} a_1' & b_1' \\ c_1' & d_1' \end{vmatrix} = \Delta^2 \begin{vmatrix} a_1 & b_1 \\ c_1 & d_1 \end{vmatrix}. \qquad (30)$$

The value of k_1, the cross-ratio of S', is by (15)

$$k_1 = \frac{\left(a_1' + d_1' - \sqrt{(a_1' + d_1')^2 - 4(a_1'd_1' - b_1'c_1')}\right)^2}{4(a_1'd_1' - b_1'c_1')}.$$

Substituting from (29) and (30) we get

$$k_1 = \frac{\left(a_1 + d_1 - \sqrt{(a_1 + d_1)^2 - 4(a_1d_1 - b_1c_1)}\right)^2}{4(a_1d_1 - b_1c_1)} = k.$$

If the value of A, one of the invariant points of S, be substituted for x in the equation, $x_1 = \frac{ax+b}{cx+d}$, x_1 will be found to be equal to A_1, an invariant point of S'. Thus T transforms the invariant points of S into those of S'.

THEOREM 11. When T operates on S to produce S', according to the formula $S' = T^{-1}ST$, the invariant points of S are transformed by T into the invariant points of S' and the cross-ratio of S' is the same as that of S.

§ 3. One-Parameter Groups of Projective Transformations.

26. *Resultant of T and T_1 with Common Invariant Points.* Let T and T_1 be two transformations of type I having the same invariant points A and A', and let T transform the point x to x_1, and let T_1 transform x_1 to x_2. The resultant of T and T_1 also leaves A and A' invariant and transforms x directly to x_2. Let T and T_1 be given in the implicit normal forms:

$$\frac{x_1 - A}{x_1 - A'} = k\frac{x - A}{x - A'} \quad \text{and} \quad \frac{x_2 - A}{x_2 - A'} = k_1\frac{x_1 - A}{x_1 - A'}. \qquad (14)$$

We eliminate x_1 from these equations by multiplication, and obtain T_2:

$$\frac{x_2 - A}{x_2 - A'} = kk_1\frac{x - A}{x - A'}.$$

The cross-ratio of T_2 is therefore $k_2 = kk_1$. Since $kk_1 = k_1k$, it follows that T and T_1 are commutative, thus completing theorem 7.

In the same way it may be shown that the resultant of any number of transformations with the same invariant points has its cross-ratio equal to the continued product of the cross-ratios of the components.

The cross-ratio k is a complex number and may have a doubly infinite number of values; hence there are a doubly infinite number of transformations, leaving two given points A and A' invariant. In fact the system of transformations leaving A and A' invariant contains a transformation corresponding to each number of the complex number system. Certain transformations of this system, corresponding to certain special values of k, have received special names. Thus the transformations of the system corresponding to $k = 1, -1, 0, \infty$ are called the identical, the involutoric, and the two pseudo-transformations, respectively. Any transformation of the system for which $k = 1 + \delta$, where δ is an infinitesimal, is called an infinitesimal transformation.

The transformations of this system have the property that the resultant of any two of them is a transformation of the same system; and the inverse of every transformation of the system is also in the system, as is shown below. Every system of transformations having these two properties is called a *group* of transformations. This group of transformations leaving A and A' invariant is called a continuous group, since continuous variation of k gives rise to transformations, all of which belong to the group. This group is designated by the symbol $G_1(AA')$ and is called a one-parameter group, the cross-ratio k being the variable parameter of the group.

THEOREM 12. The totality of projective transformations which leave the same two points of a line invariant forms a continuous group; the cross-ratio of the resultant of any two transformations of this group is equal to the product of the cross-ratios of the components.

27. *Properties of the Group* $G_1(AA')$. The fundamental property of the group $G_1(AA')$ is that the resultant of any two transformations of the group is another of the same group. This is called the *first group property.* Other properties of the group will now be developed.

The inverse of T, any transformation in $G_1(AA')$, is also to be found in $G_1(AA')$. To show this let T be the transformation which transforms the point x into x_1; then T is given by the equation

$$\frac{x_1 - A}{x_1 - A'} = k\,\frac{x - A}{x - A'}. \tag{14}$$

The inverse of T transforms x_1 back into x; then T^{-1} is given by

$$\frac{x - A}{x - A'} = \frac{1}{k}\,\frac{x_1 - A}{x_1 - A'}.$$

Hence the cross-ratio of the inverse of T is given by $1/k$; in other words the cross-ratios of a pair of inverse transformations have reciprocal values. Since k is any number in the complex number system, its reciprocal, $1/k$, is also a number in the same system. Hence the inverse of every transformation in the group $G_1(AA')$ is also in the group. This is called the *second group property.*

The resultant of a pair of inverse transformations is the identical transformation, whose cross-ratio is given by $k \times 1/k = 1$. Hence the group $G_1(AA')$ contains the identical transformation.

The group $G_1(AA')$ contains one transformation which is identical with its own inverse. In this case we have the condition $k = \frac{1}{k}$, or $k^2 = 1$; whence $k = \pm 1$. The value $k = 1$ gives the identical transformation of the group. That this is its own inverse is self-evident. The value $k = -1$ gives the involutoric transformation of the group. This transformation has the effect of interchanging every pair of corresponding points on the line, since its second power is the identical transformation; thus this transformation gives rise to an involution, whence its name.

The group $G_1(AA')$ contains two very noteworthy transformations whose cross-ratios are 0 and ∞, respectively. The first transforms all points of the line except A' into A; the second transforms all points of the line except A into A'. These are pseudo-transformations and may be regarded as forming an inverse pair.

The cross-ratio of the identical transformation is unity, and this transformation leaves every point of the line invariant. The transformation of the group whose cross-ratio is $1 + \delta$, where δ is an infinitesimal number, moves every point on the line an infinitesimal distance, and hence is called an infinitesimal transformation. δ has an infinite number of different values, viz., $|\rho| e^{i\theta}$, where ρ is an infintesmal and θ varies from 0 to 2π. If an infinitesimal transformation be repeated n times, the cross-ratio of the resultant is $(1 + \delta)^n$. By a proper choice of δ, *i. e.*, of θ, and of n (sufficiently large), this cross-ratio may be made any number we please; hence every transformation in $G_1(AA')$ may be generated from an infinitesimal transformation of the group. The chief properties of the group $G_1(AA')$ may be summed up as follows:

THEOREM 13. The resultant of any two transformations of the group $G_1(AA')$ is a third transformation of the same group; the transformations of the group can be arranged in inverse pairs; it contains the identical, one involutoric, two pseudo, and an infinite number of infinitesimal transformations; every transformation of the group can be generated from an infinitesimal transformation of the group.

28. *One-Parameter Group $G_1'(A)$.* Let T and T_1 be two transformations of type II having the same invariant point A. They may be written:

$$\frac{1}{x_1-A}=\frac{1}{x-A}+t \text{ and } \frac{1}{x_2-A}=\frac{1}{x_1-A}+t_1. \qquad (18)$$

T transforms x to x_1, and T_1 transforms x_1 to x_2. Their resultant T_2 is obtained by eliminating x_1 from these two equations by addition, giving us:

$$\frac{1}{x_2-A}=\frac{1}{x-A}+t+t_1.$$

Thus, $t_2=t+t_1$. The resultant, T_2, is of type II (thus completing theorem 10), has the same invariant point A, and its constant, t_2, is equal to the sum of the constants of T and T_1.

The parameter, t, being a complex number, may have any one of a doubly infinite number of values; and hence there are a doubly infinite number of transformations of type II having the same invariant point. This system of transformations of type II having the same invariant point possesses the first group property, as has just been shown. That it also possesses the second group property we proceed to show. Let T be the transformation:

$$\frac{1}{x_1-A}=\frac{1}{x-A}+t; \qquad (18)$$

its inverse, T^{-1}, which transforms x_1 back to x, is

$$\frac{1}{x-A}=\frac{1}{x_1-A}-t\,.$$

Hence the parameters of a pair of inverse transformations are numerically equal but of opposite signs. Since the negative of every complex number is also a complex number, it follows that the inverse of every transformation in the system is also in the system. Therefore the system of transformations of type II having the same invariant point possesses the second group property. This system has both of the defining group properties and is therefore a group. This group is continuous; it contains a transformation for every value t of the complex number system. It is designated by $G_1'(A)$.

THEOREM 14. **The totality of transformations of type II which leave the same point invariant forms a continuous group; the constant, t, of the resultant of any two transformations of the group is equal to the sum of the constants of the components.**

29. *Properties of the Group* $G_1'(A)$. The resultant of a pair of inverse transformations is the identical transformation whose constant is $t_2=t-t=0$. The group $G_1'(A)$ therefore contains the identical transformation.

The only transformation in the group which is its own inverse is the identical transformation, *i. e.*, the group contains no involutoric transformation. It contains one pseudo-transformation for which $t=\infty$. This transforms every point on the line to the invariant point.

A transformation of the group whose constant t is infinitesimally near to zero, *i. e.*, $t=|\rho|e^{i\theta}$, where ρ is an infinitesimal and θ varies from 0 to 2π, is an infinitesimal transformation. If an infinitesimal transformation is repeated n times, the resultant has the constant nt. By a proper choice of n and θ this may be made any number we please; hence every transformation in the group $G_1'(A)$ can be generated from an infinitesimal transformation of the group.

THEOREM 15. **The resultant of any two transformations of the group $G_1'(A)$ is also a transformation of the group; its transforma-**

tions can be arranged in inverse pairs; it contains the identical transformation, one pseudo, but no involutoric transformation; it contains an infinite number of infinitesimal transformations, and every transformation of the group can be generated from an infinitesimal transformation of the group.

30. *Number of One-Parameter Groups.* We have thus found two types of one-parameter groups of transformations of the points on a line, viz., $G_1(AA')$ and $G_1'(A)$. Evidently there are as many groups of the first type as there are pairs of points on a line, viz., ∞^2. Also, there is a group of type II for every point on a line; therefore, ∞^1 in number. It is also evident that every transformation of the points on the line belongs to one and only one of these one-parameter groups (except the identical transformation which is common to all).

§4. Two- and Three-Parameter Groups of Projective Transformations.

We shall now investigate the question of the existence of two-parameter groups of projective transformations of points on a line. We shall make use of a method which is of great importance and will be often used in the following chapters to prove the existence of groups of transformations.

31. *The Group $G_2(A')$.* We wish to examine the aggregate of transformations which leave a single point invariant. Let us take two transformations, T and T_1, having one, but only one, invariant point A' in common. The point A' may be taken for the origin without loss of generality. Let T and T_1 be taken in the normal form,

$$T: \quad x_1 = \frac{\begin{vmatrix} x & 1 & 0 \\ A & 1 & A \\ A' & 1 & kA' \end{vmatrix}}{\begin{vmatrix} x & 1 & 0 \\ A & 1 & 1 \\ A' & 1 & k \end{vmatrix}}; \qquad T_1: \quad x_2 = \frac{\begin{vmatrix} x_1 & 1 & 0 \\ A_1 & 1 & A_1 \\ A_1' & 1 & k_1A_1' \end{vmatrix}}{\begin{vmatrix} x_1 & 1 & 0 \\ A_1 & 1 & 1 \\ A_1' & 1 & k_1 \end{vmatrix}}. \qquad (22)$$

Making $A' = 0$ and $A_1' = 0$ in (22) these simplify to

$$x_1 = \frac{x}{\left(\frac{1-k}{A}\right)x + k} \text{ and } x_2 = \frac{x_1}{\left(\frac{1-k_1}{A_1}\right)x_1 + k_1}. \tag{31}$$

Eliminating x_1 from these two equations we get

$$T_2: \quad x_2 = \frac{x}{\left(\frac{1-k_1}{A_1} + \frac{k_1\,(1-k)}{A}\right)x + k\,k_1}. \tag{31a}$$

But this is the same form as (31), viz.:

$$T_2: \quad x_2 = \frac{x}{\left(\frac{1-k_2}{A_2}\right)x + k_2}. \tag{31b}$$

Comparing coefficients in (31a) and (31b), we have

$$\begin{gathered} k_2 = k k_1 \\ \frac{1-k_2}{A_2} = \frac{1-k_1}{A_1} + \frac{k_1\,(1-k)}{A}. \end{gathered} \tag{32}$$

These two equations enable us to express k_2 and A_2 in terms of k, k_1, A and A_1.

From these results we see that the resultant of two transformations of type I, having one invariant point in common, has for one of its invariant points the common invariant point of the components, in this instance the origin. The first of equations (32) shows us that the cross-ratio of the resultant is also equal to the product of the cross-ratios of the components, viz.: $k_2 = kk_1$, just as in the case where the two invariant points are common to the two transformations.

Since (31a) is of the same form as (31), we see that the *first* group property is satisfied, *i. e.*, in the set $S_2(A)$ of ∞^2 transformations given by (31) the resultant of any two of the set is also in the set. The two parameters of the set are the cross-ratio k and the abscissa of the other invariant point A. The structure of the set is evident; the origin A' may be taken in turn with every other point on the line to form the invariant points of a group $G_1(A'A)$ and once with itself to be the invariant point of $G_1'(A')$. Hence it contains ∞^1 one-parameter groups of type I and one of type II. Every transformation in the set $S_2(A)$ belongs to one of these one-parameter groups; its inverse is in the same group and hence also in S_2. The set of transformations $S_2(A)$ has therefore both group properties; (1) the resultant of two transformations of the set is in the set; (2) the inverse of every transformation in the set is also in the set. Hence the set $S_2(A)$ is a group $G_2(A)$.

THEOREM 16. All transformations which have a common invariant point form a two-parameter group; the cross-ratio of the resultant of any two transformations of the group is equal to the product of the cross-ratios of the components.

32. *Properties of* $G_2(A)$. From the continuity of the point system on a line and from the known continuity of each subgroup, we infer the continuity of the group $G_2(A)$. The transformations of the group $G_2(A)$ are not commutative. Since $k_2 = kk_1$, it is evident that the cross-ratio of the resultant is independent of the order of the components; but the position of the second invariant point of T_2 is not independent of the order of T and T_1. For if A and A_1 are interchanged in (32), the value of A_2 is changed, thus showing that T and T_1 are not commutative in $G_2(A)$.

When T and T_1 have *both* invariant points in common and $k_1 = \frac{1}{k}$, their resultant is the identical transformation (art. 27); but when T and T_1 have only *one* invariant point in common and $k_1 = \frac{1}{k}$, the resultant is of type II. For putting $k_1 = 1/k$ in (32) we get

$$\frac{1-1}{A_2} = \frac{k-1}{k}\left(\frac{1}{A_1} - \frac{1}{A}\right), \tag{33}$$

whence A_2 must equal zero, since neither factor on the right can be zero. Thus the two invariant points of T_2 coincide and it is of type II. The value of the constant t of T_2 is found as follows:

$$t_2 = \lim_{k_2=1} \frac{1-k_2}{A_2} = \frac{k-1}{k}\left(\frac{1}{A_1} - \frac{1}{A}\right). \tag{34}$$

THEOREM 17. The group $G_2(A)$ contains ∞^1 subgroups $G_1(AA')$ and one subgroup $G_1'(A)$. The transformations in $G_2(A)$ are not commutative. The resultant of two transformations of type I in $G_2(A)$, for which $k_1 = \frac{1}{k}$ and A_1' not equal to A', is of type II.

33. *The Three-Parameter* G_3. It was shown (theorem 6) that the resultant of T and T_1, any two projective transformations of the points on a line, is again a projective transformation; also, (art. 4.) that the inverse of every such transformation is a projective transformation. From this we infer that all projective transformations of the points on a line form a group. This is called the general projective group G_3. It is a group of three parameters; for the equation of T contains three independent parameters, viz., $a : b : c : d$. If these coefficients, a, b, c, d, be made to vary continuously, all the resulting transformations belong to the group G_3; and conversely all transformations belonging to the above group are obtained by continuously varying the coefficients in T. Such a group is evidently continuous. If the equation of T be put into the normal form,

$$\frac{x_1 - A}{x_1 - A'} = k\frac{x - A}{x - A'}, \tag{14}$$

the three natural parameters, A, A', k, may be made to vary continuously, thus generating the group G_3. The group G_3 contains ∞^1 two-parameter groups $G_2(A)$, one for each point on the line. It contains, as we have already shown, ∞^2 groups $G_1(AA')$ and ∞^1 groups $G_1'(A)$.

34. *The Mixed Group* $mG_1(AA')$. The one-parameter continuous group, $G_1(AA')$, is made up of transformations, each of which leaves the points A and A' separately invariant. The points A and A' may be interchanged by certain transformations of the points on the line. The aggregate of all transformations, which leave the pair of points AA' invariant, either separately or by interchanging them, is called the mixed group, $mG_1(AA')$.

The only transformation of the points on a line interchanging a pair of points is an involutoric transformation. Let the four points A', A, P, Q form a harmonic range and let T be the involutoric transformation of the group $G_1(PQ)$. T will interchange A' and A. Since there are ∞^1 pairs of points that divide A' and A harmonically, it follows that there are ∞^1 involutoric transformations that interchange A' and A.

The system of transformations in $mG_1(AA')$ possesses both group properties. This is known to be true for these transformations in $mG_1(AA')$ which belong to the continuous group $G_1(AA')$; but it must be proved for those transformations that interchange A' and A. Let T and T_1 be two involutoric transformations each interchanging A' and A; their resultant, therefore, leaves both A and A' separately invariant, and hence belongs to the continuous group $G_1(AA')$ and is also in $mG_1(AA')$. Let T and T' be two transformations, the first leaving A' and A separately invariant, and the second interchanging A' and A. Their resultant interchanges A' and A, and is therefore an involutoric transformation belonging to $mG_1(AA')$. Hence all transformations in $mG_1(AA')$ have the first group property. Since every involutoric transformation is its own inverse (art. 27), it follows that $mG_1(AA')$ has the second group property. Hence it is appropriate to call the set of transformations in $mG_1(AA')$ a mixed group.

THEOREM 18. The aggregate of those transformations interchanging a pair of points and those leaving them separately invariant forms a mixed group $mG_1(AA')$.

35. *Operation by T on G.* If we operate with T as in art. 24 on all the transformations of a group G, we produce thereby a new group G'. This is proved as follows: Let S and S_1 be any two transformations of G and let $SS_1 = S_2$, whence S_2 is also a transformation in G. Operating with T on S and S_1 we get

$$S' = T^{-1}ST \text{ and } S_1' = T^{-1}S_1T;$$

hence $$S'S_1' = T^{-1}STT^{-1}S_1T = T^{-1}SS_1T = T^{-1}S_2T,$$

or $$S_2' = T^{-1}S_2T,$$

i. e., the result of operating with T on the resultant of S and S_1, is the resultant of S' and S_1'. Therefore if the transformations S form a group G, the transformations S' form a new group G'. We express this by saying that T has transformed the group G into G'.

If T belongs to G, it is evident that G' is the same group as G; for the resultant of T and S is always a transformation in G and the resultant of T^{-1} and (ST) is also in G. If T does not belong to G, then G and G' are usually not the same group, but may be the same in some cases.

If G is the general projective group in one dimension and T is also a projective transformation, then G' is the general projective group. If G is some subgroup of the general projective group and T does not belong to G, then the transformed groups G' and G are said to be equivalent subgroups. The invariant figure of G is tranformed by T into the invariant figure of G' and corresponding transformations in G and G' have the same cross-ratio; the two subgroups G and G' have, therefore, the same structure.

36. *Invariant Subgroup.* When a group G is transformed by T into G', the subgroups of G go over into the subgroups of G'. When T belongs to G, the subgroups of G are only interchanged, since G' is the same as G. If a subgroup of G not containing T is transformed by T into itself, such a subgroup is called an invariant subgroup of G.

As an example let us operate on the group $G_2(A)$ by any transformation T belonging to $G_2(A)$. Take S in the form $x_1 = \frac{a_1 x}{c_1 x + d_1}$ and T in the same form $x_1 = \frac{ax}{cx + d}$. S' is readily found by making $b = b_1 = 0$, in equation (28). Thus

$$S': \quad x_1 = \frac{aa_1 x}{[c(a_1 - d_1) + dc_1]x_1 + ad_1}. \tag{35}$$

Since S' is also in $G_2(A)$, it follows that T has transformed $G_2(A)$ into itself.

$G_2(A)$ contains one subgroup of type II, viz., $G_1'(A)$. If S be chosen from this group, its equation becomes $x_1 = \frac{a_1 x}{c_1 x + a_1}$. S' then reduces to $x_1 = \frac{aa_1 x}{dc_1 x + aa_1}$, whence S' also belongs to $G_1'(A)$. Hence $G_1'(A)$ is an invariant subgroup of $G_2'(A)$.

37. *Transformations of Pencils of Lines and Planes.* The theory sketched in the foregoing pages applies equally well to the one-dimensional transformations of the lines of a flat pencil or the planes of an axial pencil. There are two varieties of such transformations, viz., those with two invariant elements and those with only one invariant element.

In the first case let O be the vertex of a flat pencil, A and A' the two invariant lines of the pencil, and x and x_1 any pair of corresponding lines in the transformation. Then we have the cross-ratio $O(A'Axx_1) = k$, and the theory requires no further development.

The second case, with one invariant element, may be deduced as the limiting form of the first case in the following manner: Let $O(A'Axx_1) = k$; whence $O(A'xAx_1) = 1 - k$. Writing out the last cross-ratio in full, we have:

$$\frac{\sin(AOA')}{\sin(AOx)} : \frac{\sin(x_1OA')}{\sin(x_1Ox)} = 1 - k.$$

Whence

$$\frac{\sin(x_1Ox)}{\sin(AOx) \cdot \sin(x_1OA')} = \frac{1-k}{\sin(AOA')}.$$

But $(x_1Ox) = (A'Ox) - (A'Ox_1)$; therefore,

$$\lim_{A'=A} \frac{\sin(A'Ox)\cos(A'Ox_1) - \cos(A'Ox)\sin(A'Ox_1)}{\sin(AOx) \cdot \sin(x_1OA')} = \lim_{A'=A} \frac{1-k}{\sin(AOA')} = t.$$

Hence $$\cot(x_1OA') - \cot(xOA') = t,$$

or $$\cot\theta_1 = \cot\theta + t. \qquad (36)$$

THEOREM 19. In a transformation of a pencil of lines (or planes) of type II, the difference of the contagents of the angles made with the invariant line (or plane) by a pair of corresponding lines (or planes) is constant for all pairs of corresponding lines (or planes).

§ 5. Projective Transformations.

38. The theory developed in the preceding §§ 1–4 is perfect, complete, and perfectly general. The special case of greatest interest is that in which the variables x and x_1 and the constants a, b, c, d in the equation,

$$x_1 = \frac{ax+b}{cx+d}, \tag{1}$$

are real numbers. Such a transformation transforms real points into real points; for if real values of x are substituted in equation (1), a, b, c, d being real, then x_1 is also real.

The theory as developed in § 1 is modified in the case of real transformations in only one particular, viz.: in regard to the invariant points of the transformations. The invariant points of the transformation (1) are given by the roots of the quadratic equation

$$cx^2 + (d-a)\,x - b = 0\,.$$

With real coefficients the roots of this equation are real and unequal, real and equal, or conjugate imaginary, according as $(a+d)^2 - 4(ad-bc) \overset{>}{\underset{<}{=}} 0$. There are thus three kinds of real projective transformations of the points on a line, distinguished by the character of the invariant points. When the invariant points of the transformation are real and distinct, it is called a *hyperbolic* transformation; when they are coincident, it is called *parabolic;* when they are conjugate imaginary, it is called *elliptic.*

The character of the cross-ratio k is also different in the hyperbolic and elliptic cases. From equations (13) and (15) it follows that k is real when A and A' are real; and complex, when A and A' are conjugate imaginary. It follows also from equation (15) that in the elliptic case, k is a complex number, and $|k| = 1$, *i. e.*, $k = e^{i\theta}$, where

$$\theta = 2\text{arc cos}\,\frac{a+d}{2\sqrt{ad-bc}} = 2\text{arc cos}\,\frac{1+k}{2\sqrt{k}}\,.$$

From the fact that there are three varieties of real projective transformations on a line we may safely infer that there are three varieties of one-parameter groups of such transformations, viz.: hyperbolic, elliptic, and parabolic groups. These three types of groups must be studied separately.

39. *The Hyperbolic Group* $hG_1(AA')$. The hyperbolic group of one parameter, which is designated by the symbol $hG_1(AA')$, consists of all hyperbolic transformations which have the same pair of real invariant points, A and A', but different real cross-ratios, k. The group $hG_1(AA')$ contains a transformation corresponding to each value of k in the real number system. Hence the group contains an identical transformation for which $k = 1$, an involutoric transformation for which $k = -1$, two pseudo-transformations for which $k = 0$ and $k = \infty$, two infinitesimal transformations for which $k = 1 + \delta$ and $k = 1 - \delta$.

From the law of the combination of the cross-ratios in the group, viz.: $k_2 = kk_1$, we learn that the group $hG_1(AA')$ contains three distinct subdivisions. Subdivision I consists of all transformations for which k is between 0 and 1; subdivision II, of all for which k is between 1 and ∞; subdivision III, of all for which k is negative. The pseudo-transformation, $k = 0$, separates subdivision III from I; the identical transformation, $k = 1$, separates I from II; the other pseudo-transformation, $k = \infty$, separates II from III.

The combination of any two transformations of subdivision I gives rise to a transformation belonging to the same subdivision; for the product of two positive proper fractions is a positive proper fraction. The inverses of all transformations in subdivision I are in II. The combination of any two transformations in II gives also a transformation in II; but the inverses of those in II are in I. The combination of any two transformations in III gives one either in I or II. The involutoric transformation divides subdivision III into two parts; all the transformations in one of these parts are the inverses of those in the other part. Subdivisions I and II contain each an infinitesimal transformation. All transformations in I may be generated by repetitions of the infinitesimal transformation $T_{1-\delta}$; all transformations in II can be generated by repetitions of the other infinitesimal transformation $T_{1+\delta}$. The transformations in III, for which k is negative, cannot be generated from either infinitesimal transformation of the group.

THEOREM 20. The hyperbolic group $hG_1(AA')$ contains one identical, one involutoric, two pseudo, and two infinitesimal transformations; it consists of three subdivisions: subdivisions I and II contain each its generating infinitesimal transformation; the transformations in subdivision III cannot be generated from either infinitesimal transformation of the group.

40. *The Elliptic Group $eG_1(AA')$.* The one-parameter elliptic group, designated by the symbol $eG_1(AA')$, consists of all real transformations having the same pair of conjugate imaginary invariant points A and A'. The parameter, $k = e^{i\theta}$, is a complex number and its variation in the complex plane is confined to the unit circle about the origin. The group contains a transformation corresponding to each point on the unit circle. This circle cuts the axis of reals in only two points, viz.: when $k = 1$ and $k = -1$; hence the group contains only two transformations for which k is real. These are respectively the identical and the involutoric transformations of the group. Since $e^{i\theta}$ cannot assume either value 0 or ∞, it follows that the group contains no pseudo-transformations. The group contains two infinitesimal transformations, for which $k = e^{i\delta}$ and $k = e^{-i\delta}$, where δ is an infinitesimal.

The elliptic group $eG_1(AA')$ contains two subdivisions; subdivision I consists of all transformations in the group for which θ is positive between 0 and π; subdivision II, of all for which θ is negative between 0 and $-\pi$. The identical and the involutoric transformations of the group form the boundaries of these subdivisions. The transformations of one subdivision are the inverses of those in the other. Each subdivision contains an infinitesimal transformation. If either infinitesimal transformation be repeated n times the cross-ratio of the resultant is given by $k=e^{\pm in\delta}$; by a proper choice of n this may be made any transformation of the group for which θ is finite. Let $\theta'=2\pi-\theta$; since $e^{i\theta'}=e^{i(2\pi-\theta)}=e^{2\pi i-i\theta}=e^{-i\theta}$, it follows that any transformation of the elliptic group $eG_1(AA')$ may be generated by repeating either infinitesimal transformation of the group.

THEOREM 21. The one-parameter group $eG_1(AA')$ contains one identical, one involutoric, two infinitesimal transformations; it consists of two subdivisions; the group may be generated by either of its infinitesimal transformations.

41. *The Parabolic Group $pG_1(A)$.* All real parabolic transformations of the points on a line which have the same invariant point A form a one-parameter parabolic group, designated by $pG_1(A)$. The parameter of the group is t and the law of combination of parameters in the group is expressed by $t_2=t+t_1$. This group contains a transformation corresponding to each number in the real number system. The identical transformation of the group is given by $t=0$; the transformation corresponding to $t=\infty$ is a pseudo-transformation of the group.

This group contains two subdivisions: Subdivision I contains all transformations for which t is positive; subdivision II, all for which t is negative. The boundaries of the two subdivisions are the identical and the pseudo-transformations. The resultant of two transformations belonging to the same subdivision is a transformation belonging to that subdivision. The inverse of every transformation in one subdivision is a transformation in the other subdivision.

The parabolic group $pG_1(A)$ contains two infinitesimal transformations, viz.: those corresponding to $t=\pm\delta$, where δ is an infinitesimal. Each subdivision of the group contains an infinitesimal transformation; and each subdivision may be generated by its infinitesimal transformation, but not by the infinitesimal transformation belonging to the other subdivision. The group $pG_1(A)$ contains no involutoric transformations.

THEOREM 22. The one-parameter parabolic group of real transformations on a line contains one identical, one pseudo, and two infinitesimal transformations, but no involutoric transformation; it consists of two subdivisions each of which is generated by its own infinitesimal transformation.

42. *The Group $G_2(A)$.* The theory developed in §4 for complex constants and variables holds also for real transformations. Equations 31–34 inclusive may be interpreted in real transformations as follows. The resultant of two hyperbolic transformations with one invariant point in common is generally a hyperbolic transformation having one of its invariant points at the common invariant point of its components; and the cross-ratio of this resultant equals the product of the cross-ratios of the components. Thus the resultant of $hT(AA')$ and $hT_1(AA'')$ is $hT_2(AA''')$ and $k_2 = kk_1$. The resultant will be parabolic in case k and k_1 have reciprocal values and T and T_1 are from different one-parameter groups. (See equation 33.)

The ∞^2 real transformations leaving A invariant is made up of ∞^1 one-parameter hyperbolic subgroups, $hG_1(AA')$, where A' is in turn every point on the line except A, and one parabolic subgroup, $pG_1(A)$, the limiting case of $hG_1(AA')$ when A' coincides with A. There are no elliptic transformations leaving A invariant. These ∞^2 transformations leaving A invariant form a two-parameter group $G_2(A)$.

THEOREM 23. The group $G_2(A)$ contains ∞^1 hyperbolic subgroups $hG_1(AA')$, one parabolic subgroup $pG_1(A)$, but no elliptic transformations.

43. *The Group G_3.* The aggregate of all real transformations of the points on a line forms a three-parameter group, designated by G_3. It contains ∞^2 one-parameter hyperbolic subgroups, one for each pair of real points on the line; it con-

tains ∞^2 one-parameter elliptic subgroups, one for each pair of conjugate imaginary points on the line; it contains ∞^1 one-parameter parabolic subgroups, one for each real point on the line; it contains ∞^1 two-parameter groups, $G_2(A)$, one for each real point on the line. The structure of G_3 may be represented by the formula

$$G_3 \equiv \infty^1 G_2(A) \equiv \infty^2 hG_1(AA') + \infty^2 eG_1(AA') + \infty^1 pG_1(A).$$

§6. Theory of Projection.

Definitions. We begin with a few definitions of the terms which will be frequently used in this section. A set or row of points on a line is called a *range* of points; the line on which the points are situated is called the *base* of the range. A set of lines lying in a plane and passing through a fixed point is called a *pencil* of lines; the fixed point is called the *vertex* of the pencil, and each line of the pencil is called a *ray*.

44. *Perspective Projection.* Let a range of points, A, B, C, D. . ., Fig. 1, be given on a line l; let lines be drawn to A, B, C, D. . . ., from a point P not on the line l; these lines form

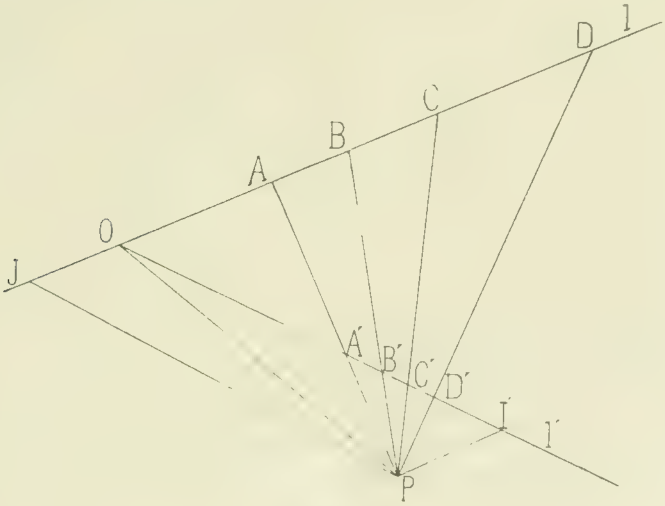

FIG. 1.

a pencil with its vertex at P. Let this pencil be cut by any other line, as l', in points A', B', C', D'... The operation of constructing the pencil through the point P and the points of the range on l is called projecting the range from P. The operation of cutting the pencil by another line, as l', is called taking a section of the pencil. The new range A', B', C', D'... on the line l' is called a perspective projection of the former range, P being the center or vertex of the projection.

45. *One-to-one Correspondence.* The points A and A', B and B', etc., are called corresponding points of the two ranges on l and l'. It is evident that to a point such as A on the line l there corresponds one and only one point, A', on l', the corresponding point lying on the same ray through P. This is true of every point on l except the point J, where the parallel to l' through P cuts l, and infinitely distant points on l. Therefore, with the exception of these points there is a one-to-one correspondence between the points of these two ranges.

46. *The Point at Infinity.* In order to make this one-to-one correspondence hold without any exceptions we adopt the following convention. We say that two parallel lines meet in one infinitely distant point. According to Euclid's hypothesis PJ is the only ray through P parallel to l'; J is therefore the only point on l which corresponds to points at infinity on l'. In the same way I' is the only point on l' which corresponds to points at infinity on l. One-to-one correspondence of points on the two ranges is therefore general with no exceptions, if we assume but a single point at infinity on each of the lines l and l'.

47. *Self-corresponding Point.* We also see that the point O on l corresponds to the point O on l'; in other words, the point of intersection of the lines l and l' is a self-corresponding point on the two ranges. Two ranges connected by a perspective projection are characterized by the facts that they have a self-corresponding point and that the rays joining corresponding points meet in a point. They are sometimes

called perspective ranges, or are said to be in perspective position.

THEOREM 24. Two ranges in perspective position have a one-to-one correspondence; the lines joining corresponding points meet in a point, the center of the perspective projection; the point of intersection of the two lines is a self-corresponding point on the two ranges.

48. *Invariance of Cross-ratios.* The cross-ratio of the four points A, B, C, D is defined by the function

$$k \equiv (ABCD) = \frac{AC}{BC} : \frac{AD}{BD}.$$

The cross-ratio of the four corresponding points A', B', C', D' is $k' \equiv (A'B'C'D') = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}$. We wish to show that these two cross-ratios are equal.

The triangles APJ and $A'PI'$, Fig. 1, are similar; also the triangles CPJ and $C'PI'$ are similar.

$$\therefore\ JA : JP = PI' : A'I' \text{ and } JC : JP = PI' : C'I',$$

$$\therefore\ JA = \frac{JP.\ PI'}{A'I'} \qquad \text{and } JC = \frac{JP.\ PI'}{C'I'}.$$

Subtracting, we get

$$AC = JC - JA = \frac{JP.\ PI'}{A'I'.\ C'I'}(A'I' - C'I') = \frac{JP.\ PI'}{A'I'.\ C'I'} \cdot A'C'.$$

In like manner we get

$$BC = \frac{JP.\ PI'}{B'I'.\ C'I'} \cdot B'C',$$

$$AD = \frac{JP.\ PI'}{A'I'.\ D'I'} \cdot A'D',$$

$$BD = \frac{JP.\ PI'}{B'I'.\ D'I'} \cdot B'D'.$$

Dividing, we get

$$\frac{AC}{BC} : \frac{AD}{BD} = \frac{A'C'}{B'C'} : \frac{A'D'}{B'D'}. \tag{37}$$

Hence the two cross-ratios $(ABCD)$ and $(A'B'C'D')$ are equal.

THEOREM 25. When two ranges of points are related by a perspective projection the cross-ratio of any four points of one range is equal to that of their four corresponding points in the other range.

49. *Non-perspective Projection.* We now proceed to consider a more general method of projecting one range into another, which method will be shown to contain the perspective method as a special case.

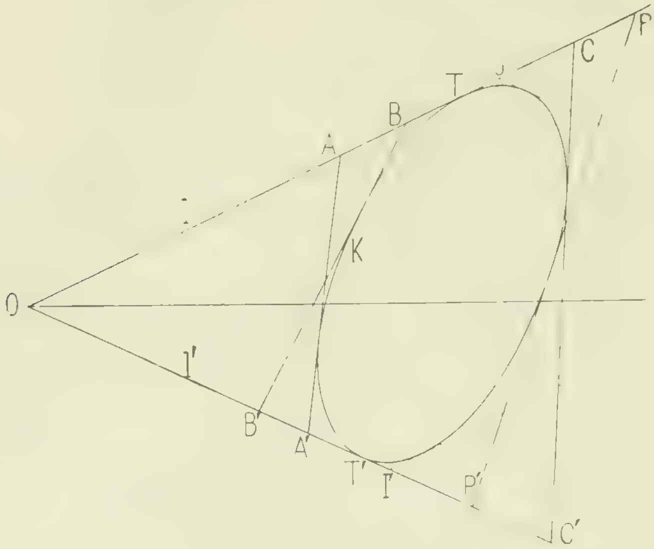

FIG. 2.

Take as before two lines l and l', Fig. 2, intersecting in O; draw any conic, for simplicity an ellipse, touching both l and l'. We shall assume as known the fundamental property of a conic that from any point outside the conic two and only two tangents can be drawn to the conic. Let P be any point on l; from P draw the two tangents to the conic K. One of them is the line l and the other intersects the line l' in some point as P'. We call P and P' corresponding points of the two ranges on l and l'. We readily see that from every point on l one and only one tangent other than l can be drawn to the conic K; and in like manner from any point on l' one and only one other tangent can be drawn to K. This construction gives a one-to-one correspondence between the points of the ranges on l and l'; and thus we see that the conic K determines a new kind of projection of the line l on l', and also of the line l' on l.

Let T and T' be the points of contact of the conic with l and l' respectively. From T only one tangent can be drawn to K and that is the line l itself; this cuts l' in O: hence T on l corresponds to O on l'. In like manner l' is the only tangent that can be drawn to K from T': hence T' on l' corresponds to O on l. The tangent to K parallel to l cuts l' in I'; hence I' corresponds to the point at infinity on l. In like manner J on l corresponds to the point at infinity on l'.

Projection of this kind is called non-perspective in order to distinguish it from the kind when the lines joining corresponding points meet in a point.

We proceed to show that perspective projection is only a special case of non-perspective projection. If the conic K touches one of the lines l or l' at O, it must also touch the other at O since it touches both. In this case the conic degenerates into a limited segment of a line having one extremity at O. Let P be any point in the plane; the segment OP, Fig. 1, may therefore be considered as a conic touching both l and l'. The tangents to this conic form a pencil of lines meeting in P; and this gives us a perspective projection with P as the center of perspective.

THEOREM 26. The non-perspective projection of one range upon another is completely determined by a conic K touching both bases. Perspective projection is a special case of non-perspective projection in which the determining conic reduces to a line-segment with one extremity at the intersection of the bases of the two ranges.

50. *Cross-ratio Unaltered by Non-perspective Projection.* We shall now prove the important fact that the cross-ratio of four points of a range is unaltered by a non-perspective projection. We shall assume the well-known theorem that any

four tangents to a conic is cut by any fifth tangent in four points whose cross-ratio is constant.* Let AA', BB', CC', PP', Fig. 2, be any four tangents to the conic K. These four tangents are cut by the tangents l and l' in the four points A, B, C, P, and A', B', C', P', respectively, whose cross-ratios are equal by the above theorem. Thus the cross-ratio of any four points is unaltered by a non-perspective projection.

THEOREM 27. If two ranges of points on intersecting lines are related by a non-perspective projection, the cross-ratio of any four points of one range is equal to that of the four corresponding points on the other range.

51. *Projective Ranges.* We have thus far defined two different methods of projecting one range of points into another, viz.: perspective and non-perspective projection. We observe that the properties of these two kinds of projection are very nearly the same. They both set up a one-to-one correspondence between the points of the two ranges, which is exceptionless when we assume a single point at infinity in each range. They both leave the cross-ratio of four points invariant. They differ only in the fact that one (perspective projection) gives a self-corresponding point in the two ranges, while the other does not. In fact, as we have shown, perspective projection is only a special case of non-perspective projection. Two ranges of points on intersecting lines are called *projective ranges*, or said to be *projectively related*, or *projective to one another* when one of them is derived from the other either by a perspective or a non-perspective projection.

Two ranges which are each projective with a third range are projective with one another. To show this let us suppose that a range R on l is projective with R_1 on l_1 and also with R_2 on l_2. Since R and R_1 are projective ranges they have a one-to-one correspondence and the cross-ratios of any four corresponding points are equal. Thus $(ABCD) = (A_1B_1C_1D_1)$. Also R and R_2 have a one-to-one correspondence and $(ABCD) =$

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* Salmon's Conic Sections, p. 252.

$(A_2B_2C_2D_2)$. Evidently R_1 and R_2 have a one-to-one correspondence and $(A_1B_1C_1D_1) = (A_2B_2C_2D_2)$. Hence R_1 and R_2 are projective.

Two projective ranges may be situated on the same line. Thus, for example, we might project perspectively a range A, B, C, D on l from two different points, P and P_1, and cut the two pencils thus formed by a second line l_1. The ranges formed on l_1 by the section of the two pencils are each projective with the given range and hence projective with each other and situated on the same line. This case is of frequent occurrence.

Two ranges of points are projective when and only when they satisfy these two conditions: first, they have a one-to-one correspondence; second, the cross-ratios of any four corresponding points are equal.

52. *Number of Points Determining Projectivity.* We now proceed to the question of the number of points on each line which it is necessary to know in order to determine the projection of one line upon the other. We shall assume the theorem that a conic is completely determined by any five independent conditions; in particular, that it is determined by any five tangents. The conic K which determines the projection of l upon l' must touch both l and l'; this gives two conditions for K. If now we select any three points on l, as A, B, C, and any three points on l', as A', B', C', to be respectively their corresponding points, the conic is completely determined; for the conic K must touch the lines AA', BB', CC', l, l'. When K is once found all pairs of corresponding points on the two lines l and l' are determined by the tangents to K. Hence three points on l and their corresponding points on l' are necessary and sufficient to determine a projection of the non-perspective kind.

In the case of perspective projection the projection is completely determined as soon as the center of the projecting pencil is known. This is determined by choosing two pairs of corresponding points on l and l' and drawing the lines joining the points of each pair. The intersection of these lines is the required center. All other pairs of corresponding points are obtained by drawing the rays of the projecting pencil. But in assuming that we have a perspective projection we assume at the same time that O is a self-corresponding point on the two lines. Thus we see, as before, that we choose three points on l and their three corresponding points on l', and thereby the projection is completely determined.

THEOREM 28. The projection of one range upon another is completely determined by three points on one range, and their three corresponding points on the other.

53. *Projective Transformation.* In Fig. 2 if l' be revolved about O until it coincides with l, any point P' on l' will be brought to some point P_1 on l, so that $OP' = OP_1$. The two ranges of points are then considered as existing on the same line l. The operation of projecting by means of the conic K a range of points on l into a new range on l' and then by revolution about O bringing the new range back to l will be called a *projective* transformation. The effect of a projective transformation is to shift the points of a line into new positions so that there is a projective relation between the old and new positions of the points.

THEOREM 29. Given two lines, l and l', intersecting at O; a projective transformation of the points on l is completely determined by means of a conic K touching both l and l'.

54. *Analytic Representation of a Projective Transformation.* We now proceed to consider the analytical aspect of a projective transformation of the points on a line. To this end we shall make use of the theorem that the cross-ratio of any four points $(ABCX)$ is equal to that of their four corresponding points $(A'B'C'X')$. Taking O as the origin, let the distances to the four points A, B, C, and X be a, b, c, and x; and let the distances to the four corresponding points be a_1, b_1, c_1, and x_1. Since the cross-ratios of these two sets of points are equal we have

$$\frac{c-a}{c-b} : \frac{x-a}{x-b} = \frac{c_1-a_1}{c_1-b_1} : \frac{x_1-a_1}{x_1-b_1}.$$

Simplifying, this reduces to the bilinear form

$$rxx_1 - px + sx_1 - q = 0\,;$$

or solving for x_1, we have

$$x_1 = \frac{px+q}{rx+s}. \qquad (1)$$

This equation represents the relation between x and x_1, any two corresponding points in a projective transformation. This equation therefore represents a projective transformation, and all properties of the transformation may be deduced analytically from the equation.

THEROEM 30. A projective transformation of the points on a line is represented analytically by a linear fractional equation in one variable. Or otherwise expressed, a linear fractional transformation in one variable is a projective transformation.

55. *Summary.* In § 1 we defined a projective transformation analytically by the linear fractional equation,

$$x = \frac{ax+b}{cx+d}, \qquad (1)$$

and proceeded to deduce the properties of projective transformations from this definition. In the present section we have defined a projective transformation geometrically, and have shown that its analytical representation is a linear fractional transformation of the form of equation (1). This proves that the transformation as defined in two entirely different ways is one and the same, and that the definitions are in harmony.

§ 7. Geometric Theory of Projective Transformations.

56. The instrument described in § 6 for constructing a projective transformation of the points on a line l by means of a conic K touching two lines l and l' can be used to establish many important properties of such transformations. While

this geometric construction is not necessary for the establishment of any part of the theory of such transformations, yet the method beautifully illustrates the theory and gives it a concrete geometrical form. We add in this section a brief outline of the geometric form of the theory, and in the exercises at the end of the chapter many problems depending on this construction.*

If the lines l and l' are real and the conic K is real, then real points are transformed into real points and the transformation is real. The transformation determined by the conic K is designated by T_k.

57. *Invariant Points.* We observe in the first place that a projective transformation T_k usually leaves two points on the line unaltered in position, for generally two tangents can be drawn to the conic K perpendicular to the bisector OX, Fig. 3; these cut l and l' in A and A', B and B' respectively.

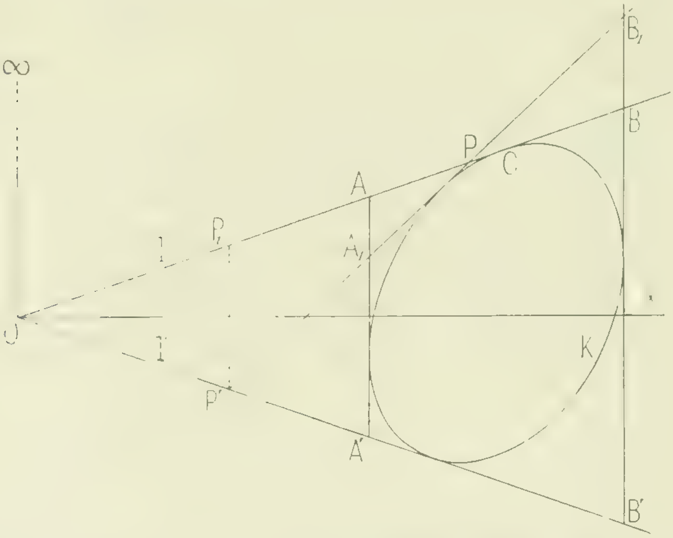

FIG. 3.

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* The whole theory of the projective transformations of the points on a line may be developed by the above geometric construction without any resort to analytical formula. It was so developed by the author in a paper entitled "Continuous Groups of Projective Transformations Treated Synthetically," published in the Kansas University Quarterly, vol. IV, No. 2, October, 1895.

A and A', B and B', are therefore corresponding points, and the revolution about O brings A' to A and B' to B. The points A and B are the *invariant* or double points of the transformation.

It has just been said that generally there are two tangents to K perpendicular to OX. This should be examined more closely. When the conic K is an ellipse, two real tangents to K can always be drawn perpendicular to OX; and hence the projective transformation determined by an ellipse always has two invariant points. When the conic K is a parabola, there are still two real tangents perpendicular to OX; but one of them is the line at infinity: hence the projective transformation determined by a parabola always has two real invariant points, one of which is the point at infinity on l.

When the conic K is a hyperbola, there are three cases to be considered. If the asymptotes of the hyperbola K make with the line OX angles which (measured in the same direction) are both less than, or both greater than, a right angle, then two real tangents to the hyperbola can be drawn perpendicular to OX, and the transformation determined by K has two real invariant points. If on the other hand the asymptotes to K make with OX angles one less than, and the other greater than, a right angle, then the tangents to K perpendicular to OX are imaginary and the transformation determined by K has its invariant points imaginary. But if K has one of its asymptotes perpendicular to OX, the transformation determined by K has one real invariant point. Or, since the asymptote to a hyperbola is the limiting position of two parallel tangents, we may say in the last case that the transformation determined by K has two coincident invariant points.

58. *Hyperbolic Transformations.* The lines AA', BB', l, l' are four fixed tangents to the conic K. Any fifth tangent, as PP', cuts these four tangents in four points whose cross-ratio is constant. The range A_1, B_1, P, P' may be projected orthogonally on l by lines drawn parallel to AA'; and the cross-

ratio $(A_1B_1PP') = (ABPP_1)$. But since the first cross-ratio is constant for all tangents to K it follows that the second is constant for all pairs of corresponding points; hence for every projective transformation which has two real invariant points we have the theorem that any pair of corresponding points and the two invariant points have a constant cross-ratio. A transformation with two real invariant points is called a *hyperbolic* transformation. In this case the constant cross-ratio k is a real number.

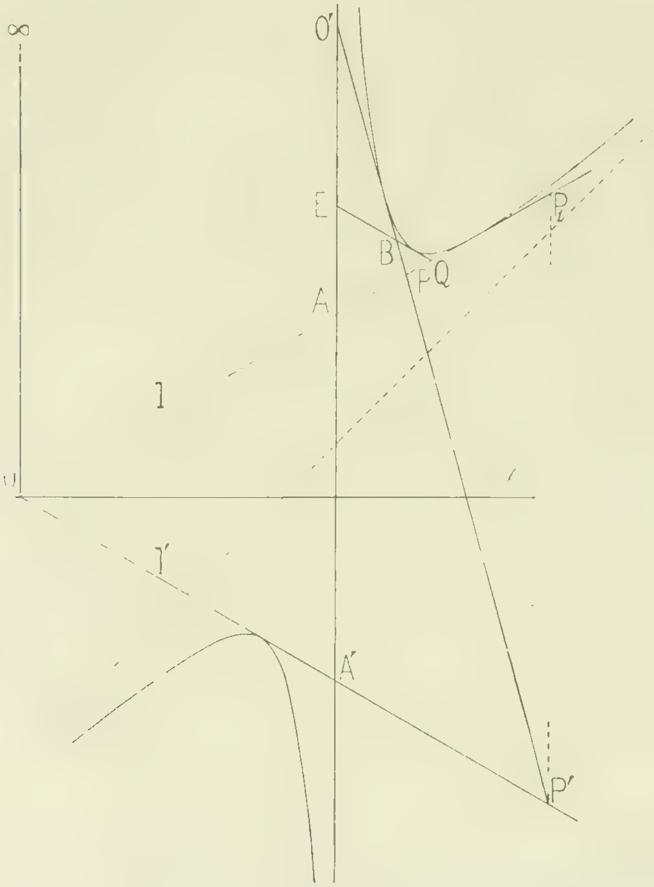

FIG. 4.

59. *Elliptic Transformations.* In the case when the two tangents to K perpendicular to OX are imaginary, it still holds that the cross-ratio of the four points of intersection of any tangent to K with the lines l, l' and the two imaginary tangents AA' and BB' is constant. The two invariant points in this case are conjugate imaginary. A transformation with two conjugate imaginary invariant points is called an *elliptic* transformation. In this case the constant cross-ratio k is a complex number of the form $e^{i\theta}$, *i. e.*, its modulus is unity. See exercise 8, page 61. This constant k in both hyperbolic and elliptic cases is called the characteristic cross-ratio of the transformation T.

60. *Parabolic Transformations.*[*] But when the invariant points of the transformation coincide, we no longer have a characteristic cross-ratio for the transformation. However, another relation is found to hold for pairs of corresponding points, which relation is constant for all pairs of corresponding points in the transformation. We shall now proceed to determine this relation. It may be obtained in a very simple manner by considering the parabolic transformation as the limiting case of a hyperbolic transformation.

We have $(ABPP_1) = k$; hence $(APBP_1) = 1 - k$. Writing this out in full we get

$$\frac{AB}{PB} : \frac{AP_1}{PP_1} = 1 - k \text{ ; hence } \frac{P_1P}{BP.\,AP_1} = \frac{1-k}{AB} .$$

When A and B coincide in $(ABPP_1) = k$, we have $k = 1$; let the limit of $\frac{1-k}{AB} = t$; then

$$\frac{P_1P}{AP.\,AP_1} = \frac{AP - AP_1}{AP.\,AP_1} = \frac{1}{AP_1} - \frac{1}{AP} = \lim \frac{1-k}{AB} = t .$$

$$\therefore \frac{1}{AP_1} = \frac{1}{AP} + t . \qquad (38)$$

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[*] The terms Hyperbolic, Elliptic, and Parabolic Transformations are due to Klein, and were first used by him in a paper entitled "Ueber die Transformation der elliptischen Functionen und die Auflösung der Gleichungen 5^{ten} Grades," Math. Annalen, Band 14, 1878. The names were suggested by the relations of the conic sections to the line at infinity. A hyperbola cuts the line at infinity in two real points, an

This gives a constant relation between P and P_1, a pair of corresponding points. This constant is of course the reciprocal of the segment AQ_1, where Q_1 is the point into which the point of infinity is transformed.

61. *Number of Transformations.* Every conic touching the lines l and l' determines a projective transformation. It is therefore possible to construct as many different transformations of the points on the line l as there are conics touching l and l'. We know that ∞^3 conics can be drawn touching any two lines; hence we infer that there are ∞^3 projective transformations of the points on a line. Among the ∞^3 conics touching l and l' are ∞^2 hyperbolas having one asymptote perpendicular to the line OX. Hence we infer that there are ∞^2 parabolic transformations each of which leaves only one point invariant.

62. *Continuous System of Transformations.* Our next object is to subdivide and to classify these ∞^3 transformations of the points on the line l. We consider first the quadrilateral $ABB'A'$ (Fig. 3). A range of ∞^1 conics may be described touching the sides of this quadrilateral. Call this range R. Each of these conics determines a hyperbolic transformation which has A and B for its invariant points. Each conic of the range R touches the line l at a different point; and every point of the line l is the point of contact of some conic of the range R. If C be the point of contact of conic K, the characteristic cross-ratio of the transformation produced by K is given by the cross-ratio of the four points A, B, C, O. The points A, B, O are fixed, while the point C varies for different conics of the range R. From the continuity of the point system on the line l, we infer the continuity of the system of ∞^1 transformations which leave A and B invariant.

63. *One-Parameter Groups.* The range of conics inscribed

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ellipse cuts it in two conjugate imaginary points, and a parabola cuts it in two coincident points. The unique appropriateness of the names is shown by the results of art. 72.

in the real quadrilateral $ABB'A'$ determines a system of transformations each of which is hyperbolic and leaves invariant the two points A and B. These ∞^1 transformations evidently constitute the hyperbolic group $hG_1(AB)$. The conics of the range touching the line l between A and B are ellipses; those touching the line l external to the segment AB are hyperbolas.

If the two sides of the quadrilateral AA' and BB' are conjugate imaginary lines, the inscribed range of conics determines an elliptic group $eG_1(AB)$ whose invariant points are the conjugate imaginary points A and B. All conics of this range are hyperbolas.

If the range of conics consists of hyperbolas having one common asymptote perpendicular to the bisector OX, (Fig. 4), the group is parabolic, $pG_1(A)$.

THEOREM 31. The range of conics inscribed in a quadrilateral consisting of the lines l and l' and a pair of lines perpendicular to the bisector of l and l' determines a one-parameter group of transformations on the line l.

64. *Resultant of Two Elliptic or Hyperbolic Transformations.* Let T_k be the transformation of the group $G_1(AB)$, which transforms P to P_1; then $k=(ABPP_1)$. Let T_{k_1} be the transformation of the same group which transforms P_1 to P_2; then $k_1=(ABP_1P_2)$. The two transformations T_k and T_{k_1} are together equivalent to a single transformation T_{k_2} of the same group which transforms P to P_2. To prove this we have

$$k=(ABPP_1)=\frac{AP}{BP}:\frac{AP_1}{BP_1} \text{ and } k_1=(ABP_1P_2)=\frac{AP_1}{BP_1}:\frac{AP_2}{BP_2}.$$

Eliminating the fraction containing P_1 from those two equations we have

$$kk_1=\frac{AP}{BP}:\frac{AP_2}{BP_2}=(ABPP_2)\,. \tag{39}$$

The conic of the range R whose tangential cross-ratio is kk_1, gives a transformation which is equivalent to the combined effect of T_k and T_{k_1}. This may be expressed symbolically by the equation $T_k T_{k_1} \equiv T_{kk_1}$. In the same way it may be shown that the combined effect of any number of transformations of the group is equivalent to some single one of the same group. Thus $T\ T_b T_c \ . \ . \ . \ T_n = T_s$ where $s = abc \ . \ . \ . \ n$. The characteristic cross-ratio of the resultant transformation is equal to the continued product of the characteristic cross-ratios of the component transformations $T_a T_b T_c \ . \ . \ . \ T_n$.

65. *Resultant of Two Parabolic Transformations.* Let T denote a transformation of the group $pG_1(A)$ which transforms P to P_1; then $\frac{1}{AP_1} - \frac{1}{AP} = t$. Also let T_1 be another transformation of the same group which transforms P_1 to P_2; then $\frac{1}{AP_2} - \frac{1}{AP_1} = t_1$. Eliminating the fraction $\frac{1}{AP_1}$ from these two equations we have $\frac{1}{AP_2} - \frac{1}{AP} = t_2$ where $t_2 = t + t_1$. In the same way it may be shown that the resultant of any number of transformations of the group $pG_1(A)$ is another transformation of the same group, and that the characteristic constant of the resultant is equal to the sum of the constants of the components, thus $t_n = t + t_1 + t_2 + \ . \ . \ . \ t_{n-1}$.

66. *The Two-Parameter Group, $G_2(A)$.* There are ∞^3 projective transformations of the points on a line, and only ∞^1 points on the line. Hence any point can be transformed into any other point on the line or into itself in ∞^2 different ways. In other words there is a system of ∞^2 transformations which leaves any point A on the line invariant. We shall proceed to show geometrically that this system of transformation has the fundamental properties of a group, *i. e.*, that the combined effect of any two or more of the transformations of the system is equivalent to some single transformation of the system, and that the inverse of every transformation in the system is also in the system. To show this we take a range of points in the line l (Fig. 5), and project it by means of a conic K into a second range on the line l'. Revolve l' about

O until it coincides with l; we thus have a second range on l. Let A and B be the invariant points of the transformation due to K. Now project this second range into a third range on l' by means of a conic K' touching AA' and CC'. If we now join the points of the first range on l with the corresponding points of the third range on l', these joins all touch a conic K'' which determines the projection of the first range into the third. This last transformation is equivalent to the combination of the other two. AA' is one of these joins; hence the transformation determined by K'' leaves the point A invariant. This same process may be extended to any number of transformations.

For the geometric proof of the second group property, see exercise 2, page 62.

67. *Cross-ratio of the Resultant in* $G_2(A)$. The transformation produced by the conic K'' has one invariant point

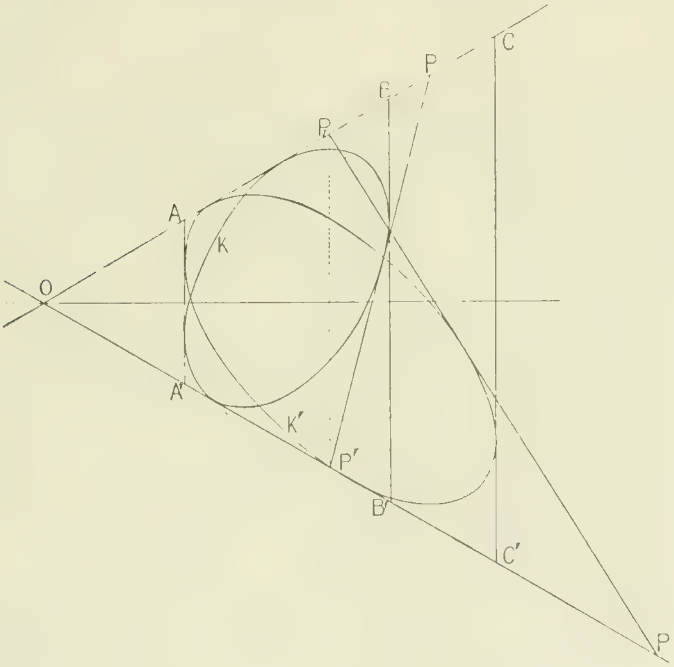

FIG. 5.

at A and another at some point D such that DD' is the tangent to K'' parallel to AA'. By means of the conic K the range $ABCD$-- is transformed into $A_1B_1C_1D_1$-, by means of the conic K' the range $A_1B_1C_1D_1$-- is transformed into A_2B_2 C_2D_2--, by means of K'' the range $ABCD$-- is transformed into $A_2B_2C_2D_2$. Consequently we have

$$(ABCD) = (A_2B_2C_2D_2).$$

But $A = A_1 = A_2$, $B = B_1$, $C_1 = C_2$, and $D = D_2$

$$\therefore\ (ABCD) = (AB_2C_1D).$$

Expanding we get

$$\frac{AC}{BC} \cdot \frac{BD}{AD} = \frac{AC_1}{B_2C_1} \cdot \frac{B_2D}{AD}, \text{ whence } \frac{AC \,.\, B_2C_1}{BC \,.\, AC_1} = \frac{B_2D}{BD}. \qquad (40)$$

The characteristic cross-ratio of the transformation T due to K is $k = (ABCC_1)$; that of the transformation T_1 due to K' is $k_1 = (AC_1BB_2)$. Expanding and multiplying:

$$kk_1 = \frac{AC}{BC} \cdot \frac{BC_1}{AC_1} \cdot \frac{AB}{C_1B} \cdot \frac{C_1B_2}{AB_2} = \frac{AC \,.\, AB \,.\, B_2C_1}{BC \,.\, AC_1 \,.\, AB_2}$$
$$= \left(\frac{AC \,.\, B_2C_1}{BC \,.\, AC_1}\right)\left(\frac{AB}{AB_2}\right).$$

Substituting from equation (39) we get

$$kk_1 = \frac{AB}{AB_2} \cdot \frac{B_2D}{BD} = \frac{AB}{DB} : \frac{AB_2}{DB_2} = (ADBB_2). \qquad (41)$$

But $(ADBB_2) = k_2$ the characteristic cross-ratio of the transformation due to the conic K'', thus $kk_1 = k_2$. *Hence the characteristic cross-ratio of the resultant in $G_2(A)$ is equal to the product of the characteristic cross-ratio of the components.*

68. *The Perspective Subgroup, G_2O.* The two-parameter subgroup $G_2(O)$, which leaves the point O invariant, is made up of one-parameter subgroups each of which leaves O and some other point, as A, invariant. Fig. 6 shows the construction of a transformation which belongs to the one-parameter subgroup $G_1(OA)$. This transformation is determined by the degenerate conic OQ. All transformations which leave the point O invariant are determined by the conics which touch l, l', and $O\infty$; but since these three lines meet in a point, it follows that all these conics must be de-

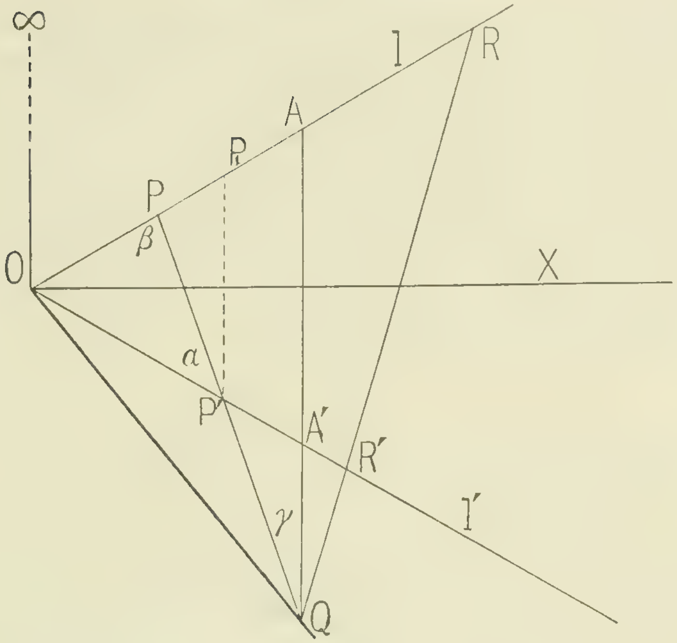

FIG. 6.

generate, and each must consist of linear segments terminating at O. Every transformation of this kind is a perspective transformation. The point Q from which the projecting lines are drawn may be any point in the plane; Q may therefore have ∞^2 different positions; and we see that there are ∞^2 perspective transformations, each of which leaves the point O invariant. These ∞^2 perspective transformations form the two-parameter group $G_2(O)$. This subgroup contains all the perspective transformations in the general projective group and no others.

THEOREM 32. **All the perspective transformations contained in the general projective group form a two-parameter subgroup $G_2(O)$; this subgroup contains no transformation which is not perspective.**

69. *Subgroups of the Perspective Group.* We now proceed to the consideration of the one-parameter subgroups which compose the two-parameter perspective group. Each of

these subgroups has the point O and some other point, as A, for the invariant points. For ∞^1 positions of the point Q the resulting perspective transformations leave the point A as well as O invariant. It is easy to see that these ∞^1 positions of Q must all be on the line AA', because the second invariant point of a perspective transformation is found by dropping a perpendicular from Q on OX. Thus the ∞^1 perspective transformations obtained by taking the center of perspective at all points on a line perpendicular to OX form a one-parameter subgroup of $G_2(O)$.

§ 8. Geometric Theory of Projective Transformations of Pencils of Lines.

We pass now to the geometric construction of projective transformation in other one-dimensional forms, viz.: in a pencil of lines through a point and a pencil of planes through a line.

70. *A Simple Construction.* A perfectly obvious construction for a projective transformation of a pencil of rays is as follows: Let two planes Pl and $P'l$ meet in a line l; and let O, a point on l, be the common vertex of two pencils of rays, one in Pl and the other on $P'l$. The planes determined by three pairs of corresponding lines of the two pencils together with Pl and $P'l$ determine a cone of the second order having its vertex at O and touching both Pl and $P'l$. Tangent planes to this cone cut Pl and $P'l$ in corresponding lines of the two pencils. If now $P'l$ be revolved about l until it coincides with Pl, a projective transformation of the pencil in Pl is completed. The properties of this transformation and groups of such transformations may easily be developed by this method.

71. *Another Method.* A second method for constructing a projective transformation of a pencil of lines is obtained by considering the process dualistic to that used for a range of

points. Let O and O' (Fig. 7) be any two points in the plane. Pass any conic K through O and O' and draw a pencil of lines through O. Join the points where these lines cut the conic K to O'. We thereby construct a pencil through O' projective

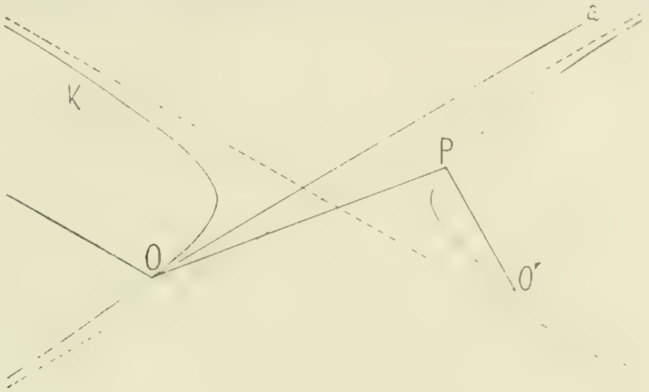

FIG. 7.

with the given pencil through O. Corresponding rays of these pencils meet on K. If the whole plane be translated along the line OO' without rotation until O' is carried to O, we then have two pencils through O which are projectively related. These operations construct a projective transformation of the pencil through O.

72. *Invariant Rays.* The transformation thus constructed usually leaves two rays of the pencil invariant; these are the rays parallel to the asymptotes of the conic K. If the conic is a hyperbola, the two invariant rays are real and the transformation is *hyperbolic*. If the conic is an ellipse, the invariant rays are conjugate imaginary and the transformation is *elliptic*. If the conic is a parabola, there is only one invariant ray, and the transformation is *parabolic*.

In the hyperbolic and elliptic transformations, the cross-ratio of the invariant rays and a pair of corresponding rays is constant for all pairs of corresponding rays. Thus cross-ratio in the first case is real; in the second case, of the form $e^{i\theta}$. No further developments are necessary in these cases.

73. *Parabolic Transformations.* The parabolic case requires special attention, for the characteristic constant t of a parabolic transformation is not subject to a dualistic interpretation. Let a transformation T be determined by a parabola K passing through O and O' (Fig. 8). The single ray left invariant by the transformation is parallel to the axis of the parabola K. Let r and r' be a pair of corresponding rays in the pencils through O and O'; and let them meet the conic K in the point Q. We make use of the following theorem for a parabola: If from any point Q on a parabola chords be drawn to O and O', two fixed points on the parabola, the difference of the cotangents of the angles which these chords make with the axis of the parabola is constant. Thus $\cot\theta_1 - \cot\theta = \cot\phi$, θ and θ_1, being the angles which the rays r and r' make with the invariant ray, and ϕ being the angle made with the invariant ray by the ray r'' which is transformed into the perpendicular to the invariant ray.

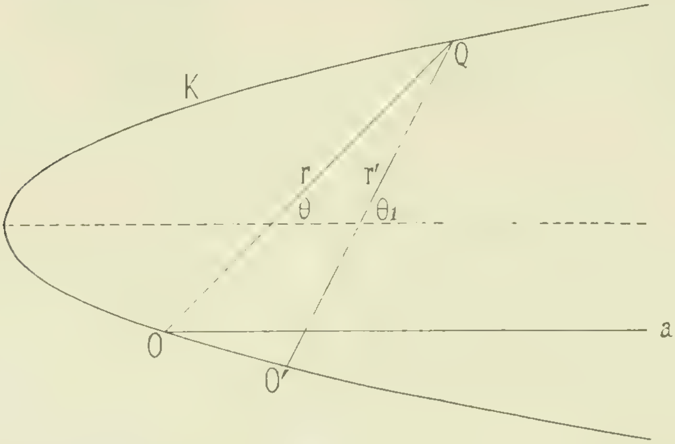

FIG. 8.

The relation $\cot\theta_1 = \cot\theta + t$, for a parabolic transformation is readily deduced as a limiting case of the cross-ratio formula of a hyperbolic transformation. (See art. 37.)

74. *Projective Transformations of a Pencil of Planes.* The theory of the real transformations of a pencil of planes is so

similar to that of a pencil of lines that nothing further than a brief statement is required on this point. The transformations are hyperbolic, elliptic, and parabolic; these groups are the same as for the other one-dimensional forms, the range of points and the pencil of lines.

Exercises on Chapter 1.

A. GENERAL ANALYTIC THEORY.

(1). Show that the determinants of a pair of inverse transformations have the same value.

(2). Show that a pair of inverse transformations are of the same type, have the same invariant points and, if of type I, reciprocal cross-ratios.

(3). Show that the resultant of a pair of inverse transformations is always the identical transformation.

(4). Show that the determinants of a pair of conjugate transformations always have the same value.

(5). Show that two conjugate transformations have the same cross-ratio but not the same invariant points.

(6). Let the invariant points of a pair of conjugate transformations be (AA') and (BB'); show that the segment (AA') is equal to the segment (BB').

(7). Show that the transformation, $x_1 = \frac{ax+b}{cx+d}$, transforms the point $x = -\frac{d}{c}$ to infinity and the point $x = -\frac{b}{a}$ to the origin. Into what points are the origin and the point at infinity transformed?

(8). Find the invariant points of the following transformations: i) $x_1 = ax + b$; ii) $x_1 = ax$; iii) $x_1 = x + b$.

(9). Find the equation of all transformations in the group G_3 that interchange two points A and A'; show that they are all involutoric transformations.

(10). Prove that the system of ∞^2 involutoric transformations in G_3 does not form a group; also the system contains no infinitesimal transformation.

(11). In a parabolic transformation show that $t = \frac{c}{\sqrt{ad - bc}}$.

(12). Prove that the ∞^2 parabolic transformation in G_3 do not form a group.

(13). Show that two transformations are commutative when, and only when, they belong to the same one-parameter group.

Let T transform S into S' according to the formula $S' = T^{-1}ST$. Then—

(14). If S is of type I, so is S'; and $k = k'$, where k and k' are the cross-ratios of S and S' respectively.

(15). If S is of type II, so is S'; find the relation between t and t', the characteristic constants of S and S' respectively.

(16). If S and T have one invariant point in common, S' has the same invariant point; in this case if S is of type II and T of type I, then $t' = kt$, where k is the cross-ratio of T.

(17). If S and T belong to the same one-parameter group, $S' = S$.

(18). Show from equations I to V, art. 21, that $\frac{1 + k_2}{\sqrt{k_2}} = \frac{(1 - k_1)(1 - k_1)l + k_1 + k_1}{\sqrt{kk_1}}$, where $l = (A_1 A A' A_1')$, the cross-ratio of the four invariant points of T and T_1.

(19). Deduce equation (*31a*) from equation (26).

(20). Deduce the canonical forms, $x_1 = kx + A(1 - k)$, $x_1 = kx$, and $x_1 = x + t$, from equation (22).

B. REAL TRANSFORMATIONS.

I. *Analytic Theory.*

(1). Find the transformation which changes the points whose coordinates are $2, 8, 9$ into the points whose coordinates are $1/2, 25/44$, and $43/75$ respectively.

$$Ans.\ (A, A') = \frac{8 \pm \sqrt{15}}{7}\,;\ k = 31 - 8\sqrt{15}\,.$$

(2). Show that every real transformation with a negative determinant is hyperbolic and its cross-ratio k is negative.

(3). Show that every transformation with positive determinant is either elliptic, parabolic, or hyperbolic with positive cross-ratio k.

(4). Show that every transformation with positive determinant can be generated by the repetition of some real infinitesimal transformation; show also that no transformation with negative determinant can be so generated.

(5). The resultant of two transformations, one with a positive and the other a negative determinant, is a hyperbolic transformation with negative cross-ratio k.

(6). Show that the resultant of two hyperbolic transformations in $G_2(A)$ is (a) parabolic when $k_1 = \frac{1}{k}$ and $A_1' \neq A'$; (b) is identical when $k_1 = \frac{1}{k}$ and $A_1' = A'$.

(7). Show that the cross-ratio of four real points on a line is always real.

(8). Show that the cross-ratio of a pair of real and a pair of conjugate imaginary points is always of the form $e^{i\theta}$.

(9). Show that the identical and the involutoric transformations divide the elliptic group eG_1 into two subdivisions each of which contains the transformations inverse to those of the other.

(10). The group eG_2 contains no pseudo-transformations.

(11). In an involutoric transformation the middle point of the segment AA' is transformed into the point at infinity.

(12). If Q is transformed to infinity and the point at infinity to Q' by a parabolic transformation T whose invariant point is A, then the points Q and Q' are equally distant from A on opposite sides.

II. *Geometric Theory.*

(1). The range of conics inscribed in the quadrilateral $ABB'A'$, Fig. 3, contains three degenerate conics, viz.: the diagonals of the quadrilateral $O\infty$, AB', $A'B$. Show (a) that the transformation determined by $O\infty$ is the identical transformation of the group $hG_2(AB)$; and (b) that the transformations determined by AB' and $A'B$ are the pseudo-transformations of the group.

(2). If the conic K' is the reflection of K on the line OX, show that the transformations T_k and $T_{k'}$ form an inverse pair.

(3). Show that the transformation determined by the conic which has the line OX for one of its axes is involutoric.

(4). What two conics determine the two infinitesimal transformations of the group $hG_1(AB)$?

(5). What conics of the range inscribed in $ABB'A'$ determine transformations belonging to subdivisions I, II, III, respectively?

(6). In the elliptic group $eG_2(AB)$, what conic determines (a) the identical, and (b) the involutoric transformation of the group?

(7). In the parabolic group $pG_2(A)$ what conic determines (a) the identical and (b) the pseudo-transformation of the group?

(8). Show that the group $G_2(A)$ contains only hyperbolic and parabolic transformations.

(9). Show that the two groups $G_2(A)$ and $G_2(A')$ have in common all the transformations of the group $hG_1(AA')$.

(10). Show that the ∞^3 conics touching l and l' determine ∞^3 transformations which form the group G_3.

(11). Show that the ∞^2 parabolas touching l and l' determine a two-parameter group; find its invariant point.

(12). Show that the system of parabolas having $O\infty$ for their common axis and touching l and l' determines a one-parameter parabolic group; find its invariant point.

(13). Let T be a transformation whose invariant points are A and ∞; and let P and P_1 be a pair of corresponding points of T; show that the characteristic cross-ratio of T is $k = AP/AP_1$.

(14). If the transformation $T(A\infty)$ transforms the segment PQ into P_1Q_1, show that the length of the segment PQ is k times the segment P_1Q_1, *i. e.*, $PQ = k(P_1Q_1)$.

This is identical with the mechanical effect of stretching a rubber cord with one end fixed at A. Such a transformation is called a Dilation, and the group $G_2(A)$ is called the group of dilations.

(15). If a tangent be drawn to one of the parabolas K of example (12) cutting l and l' at P and P', show that the difference of the segments OP and OP' is constant for all tangents to K; hence show that the transformation determined by K transforms segments into equal segments.

Such a transformation is called a Translation of the line into itself, and the group $pG_1(\infty)$ is called the group of translations.

(16). Show that the characteristic cross-ratio of a perspective transformation whose center is at Q (Fig. 9), is $k = A'Q/AQ$.

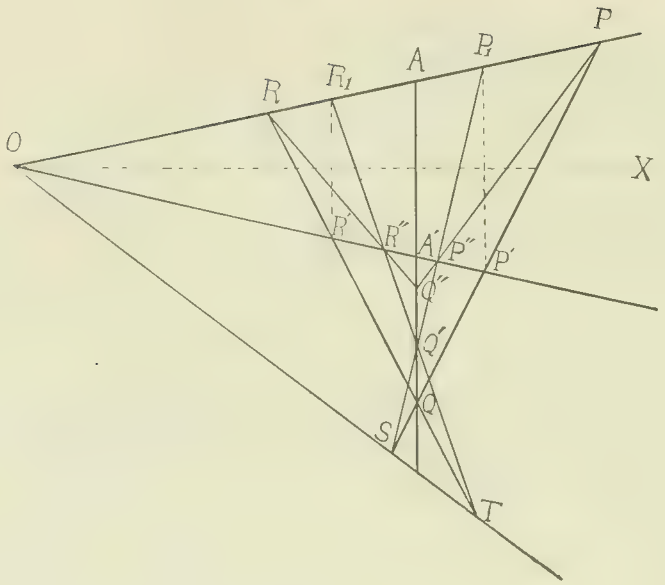

FIG. 9.

(17). In the perspective group $hG_1(AO)$ what degenerate conics determine the identical, the involutoric, the two pseudo-transformations and a pair of inverse transformations of the group?

(18). If we take on the line AA', (Fig. 9), two points Q and Q' as the centers of perspective transformations T and T_1, give a direct geometric proof that the resultant of T and T_1 is a perspective transformation whose center is also in AA'.

(19). The group $G_1(O)$ of perspective transformations contains one parabolic subgroup; find the locus of Q for this parabolic perspective group.

(20). Show that any system of parallel lines cutting l and l' determines a projective transformation T whose invariant points are O and ∞.

(21). Find the relation between the value of k in the transformation of example (20) and the slope of the system of parallel lines.

CHAPTER II.

COLLINEATIONS IN THE PLANE; TYPES AND NORMAL FORMS.

§ 1. General Analytic Theory of Plane Collineations.
§ 2. Geometric Construction of Plane Collineations.
§ 3. Types of Plane Collineations.
§ 4. Normal Forms of Equations of the Five Types.
§ 5. Canonical Forms of Equations of Collineations.
§ 6. Real Collineations.
Exercises.

75. The present chapter is devoted to the theory of projective transformations or collineations in a plane. Following the methods of the last chapter we shall first, in § 1, define a collineation analytically and develop the fundamental properties of such transformations. We shall then develop in § 2 two mutually dualistic geometric methods of constructing plane collineations. In § 3 we show both analytically and geometrically the existence of five distinct types of plane collineations. In § 4 we develop the normal forms of the defining equations of the five types, and in § 5 the canonical forms of these same equations. The special case of real collineations in the plane is then discussed in § 6, and the chapter closes with a list of exercises supplementing the theory.

§ 1. General Analytic Theory of Plane Collineations.

76. *Analytical Definition of a Plane Collineation.* Using rectangular or oblique Cartesian coordinates, the transformation of the plane which is expressed by the linear fractional equations, having the same denominator,

$$x_1 = \frac{ax+by+c}{a''x+b''y+c''} \quad \text{and} \quad y_1 = \frac{a'x+b'y+c'}{a''x+b''y+c''} \qquad (1)$$

is called a projective transformation or collineation of the plane. Using homogeneous point coordinates the same transformation is expressed by the linear equations,

$$\rho x_1 = ax + by + cz, \quad \rho y_1 = a'x + b'y + c'z, \quad \rho z_1 = a''x + b''y + c''z. \tag{2}$$

We shall generally use the Cartesian system, but may occasionally use the other form. The change from the one system to the other is so easily made that the reader will have no difficulty in passing from the one to the other at pleasure.

We shall assume throughout, unless otherwise expressly stated, that the coefficients and variables in equations (1) and (2) are complex numbers; we shall also assume that the determinant of the transformation does not vanish; thus

$$\Delta \equiv \begin{vmatrix} a & b & c \\ a' & b' & c' \\ a'' & b'' & c'' \end{vmatrix} \neq 0.$$

The reason for excluding, for the present, transformations for which the determinant vanishes will be shown later when these special transformations with determinant equal to zero will be discussed.

77. *One-to-one Correspondence of Points.* Equations (1) show that x_1 and y_1 are one-valued functions of x and y, or that to a given point (x,y) there corresponds in a collineation one and only one point (x_1,y_1). Equations (1) can be solved for x and y; the values found are

$$x = \frac{Ax_1 + A'y_1 + A''}{Cx_1 + C'y_1 + C''} \text{ and } y = \frac{Bx_1 + B'y_1 + B''}{Cx_1 + C'y_1 + C''}, \tag{3}$$

where A, A', B, etc., are the cofactors of a, a', b, etc., in Δ, the determinant of the transformation (1).

The solvability of equations (1) is secured by assuming that Δ is not zero. Equations (3) are likewise one-valued functions of x_1 and y_1. Hence to any chosen point (x_1,y_1) there corresponds one and only one point (x,y).

Equations (1) transform the point (x,y) into (x_1,y_1); while equations (3) transform the point (x_1,y_1) back to (x,y). Two such transformations are *inverse* transformations. (Arts. 3 and 27.)

78. *The Correspondence of Lines is Also One-to-one.* The transformations expressed by (1) and (3) always transform lines into lines. Let the equation of any chosen line be $lx + my + n = 0$; substitute for x and y in this equation their values from (3) and we have

$$l\left(\frac{Ax_1 + A'y_1 + A''}{Cx_1 + C'y_1 + C''}\right) + m\left(\frac{Bx_1 + B'y_1 + B''}{Cx_1 + C'y_1 + C''}\right) + n = 0. \qquad (4)$$

Clearing of fractions and collecting, we get a linear equation in x_1 and y_1, which represents a straight line. Hence the transformation (3) transforms straight lines into straight lines. In like manner the transformation (1) can be shown to transform lines into lines.

79. *A Collineation is Self-dualistic.* The transformation expressed by equations (1) is capable of a double interpretation according as the variables represent point or line coordinates. When (x, y) and (x_1, y_1) are point coordinates, equations (1) and (3) immediately show that points are transformed into points; and we are able to prove as above that lines are also transformed into lines.

On the other hand, if (x, y) and (x_1, y_1) are line coordinates, the equations show at once that lines are transformed into lines, and it can be shown by substituting in the equation of a point that points are also transformed into points.

Although we shall have but little occasion for the explicit use of line coordinates, yet the dual interpretation should be held in mind and will often be of great use to us.

THEOREM 1. A plane collineation transforms points into points and establishes a one-to-one correspondence between the points of the two configurations; it also transforms lines into lines and establishes a one-to-one correspondence between the lines of the two configurations; a plane collineation is a self-dualistic transformation.

80. *Eight Conditions Determine a Collineation.* Equations (1) contain nine coefficients; but since we may divide the numerator and denominator of both fractions through by any one of the coefficients, it follows that there are only eight independent constants. Therefore eight independent conditions are sufficient to determine a collineation. Let the coordinates of four points be (x', y'), (x'', y''), (x''', y'''), (x^{iv}, y^{iv}), and let their four corresponding points be respectively (x_1', y_1'), (x_1'', y_1''), (x_1''', y_1'''), (x_1^{iv}, y_1^{iv}). Substituting in equations (1) successively the coordinates of each pair of corresponding points we have eight equations from which to determine the eight independent constants in (1). These eight equations are linear and homogeneous in the nine coefficients a, b, c, a', etc., and therefore the eight independent constants are determined uniquely and completely.

From the principle of duality we infer that a plane collineation is also uniquely and completely determined by four lines and their four corresponding lines.

It should be understood that the four points must be so chosen that no three of them lie on a line; if four lines are chosen, no three of them pass through a point.

THEOREM 2. Any complete quadrangle or quadrilateral may be transformed into any other complete quadrangle or quadrilateral by a plane collineation in one and only one way.

81. *Cross-ratio Unaltered by a Collineation.* It was established in Chapter I, article 9, that when two lines are projectively related, the cross-ratio of any four points of the one line is equal to the cross-ratio of the four corresponding points on the other line. This fact is independent of the position of the lines. They may be coincident, they may intersect and thus lie in the same plane, or they may be non-intersecting lines in space. The same theorem is true for two projectively related pencils, and is independent of the positions of the pencils.

A plane collineation transforms a line g into g_1; the range of points on g is projectively related to the range on g_1 and the cross-ratio of any four points on g is the same as that of the four corresponding points on g_1. So also for the two pencils of lines through P and P_1, corresponding points of the transformation; the cross-ratio of four lines of the pencil through P is the same as that of their four corresponding lines through P_1.

THEOREM 3. The cross-ratio of any four collinear points or concurrent lines of the plane is unaltered by a collineation.

82. *The Line at Infinity.* The collineation expressed by equations (1) transforms a point (x, y) into (x_1, y_1). If (x, y) be a point on the line $a''x + b''y + c'' = 0$, then (x_1, y_1) is a point at infinity; for $a''x + b''y + c''$ is the common denominator of the two fractions in equations (1) and vanishes for all points (x, y) which lie on the line $a''x + b''y + c'' = 0$. Every point on this line is transformed into a point at infinity; hence the line, $a''x + b''y + c'' = 0$, is transformed into the line at infinity in the plane.*

The line $ax + by + c = 0$ is transformed into the axis $x_1 = 0$, and the line $a'x + b'y + c' = 0$ is transformed into the axis $y_1 = 0$. Hence, the triangle formed by the three lines, $ax + by + c = 0$, $a'x + b'y + c' = 0$, $a''x + b''y + c'' = 0$, is transformed into the triangle formed by the coordinate axes and the line at infinity. That the first three lines actually form a triangle is secured by the condition $\Delta \neq 0$.

83. *Invariant Points of a Collineation.* If the collineation expressed by equations (1) transforms any point (x, y) into itself, then x_1 and y_1 become x and y. The coordinates of such a point may be found by solving the equations

$$x = \frac{ax + by + c}{a''x + b''y + c''} \quad \text{and} \quad y = \frac{a'x + b'y + c'}{a''x + b''y + c''}. \tag{5}$$

Clearing of fractions these become

$$a''x^2 + (c'' - a)x - c + (b''x - b)y = 0,$$
$$\text{and } b''y^2 + (a''x + c'' - b')y - (a'x + c') = 0.$$

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*For further discussion of the line at infinity see art. 88.

Eliminating y we get

$$\begin{vmatrix} b'' & a''x + c'' - b' & -a'x - c' \\ b''x - b & a''x^2 + (c'' - a)\,x - c & 0 \\ 0 & b''x - b & a''x^2 + (c'' - a)\,x - c \end{vmatrix} = 0. \qquad (6)$$

When the determinant is expanded, the coefficient of x^4 vanishes and we have a cubic equation of the form,

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0, \qquad (7)$$

from which to find x. Let the three roots of this cubic be A, A', A''. Substituting their values in the first of equations (5), we find three values of y, viz.: B, B', B''. The three points whose coordinates are (A, B), (A', B'), (A'', B'') are invariant points of the transformation (1). In the most general case these points form a triangle which is called the invariant triangle of the transformation. There are special cases to be considered when the three invariant points do not form a triangle; for example, two of the three points may coincide, or all three may lie on a line, etc. All these special cases will be determined later.

THEOREM 4. A collineation of the most general kind leaves three linearly independent points of the plane invariant.

84. *Invariant Lines of a Collineation.* If equations (1) be interpreted in line coordinates instead of point coordinates, then the analytic work of the last article shows that a collineation in a plane leaves three lines invariant; these generally form a triangle. We have just shown that a collineation of the most general kind leaves invariant three points and three lines. The relation of these three points and three lines is evident at once. The three points are the vertices and the three lines are the sides of the same invariant triangle. Let A, B, C designate the vertices, and a, b, c the opposite sides respectively of the triangle.

If a collineation leave both A and B invariant, then c, their join, is transformed into itself; for c is transformed into some line c_1, which must pass through A and B, because they are unaltered in position; and hence c and c_1 must be the same line. It is to be understood that not every point on c is transformed into itself; this is true only of A and B. But every point on c, except A and B, is transformed into some other point also on c. Thus the points on c undergo a one-dimensional projective transformation, A and B being the invariant points of the transformation. The same holds for the other sides, a and b, of the invariant triangle. In like manner the pencils through the invariant points A, B, C undergo one-dimensional projective transformations.

THEOREM 5. A plane collineation of the most general kind leaves a triangle invariant, and produces a one-dimensional projective transformation along each of the invariant lines and through each of the invariant points.

85. *The Identical Collineation.* The question at once presents itself whether there exist collineations in the plane which leave invariant more than three points or more than three lines. Suppose we have a collineation T leaving invariant the triangle ABC and a fourth point D of the plane, such that no three of the four points are in a line (Fig. 10).

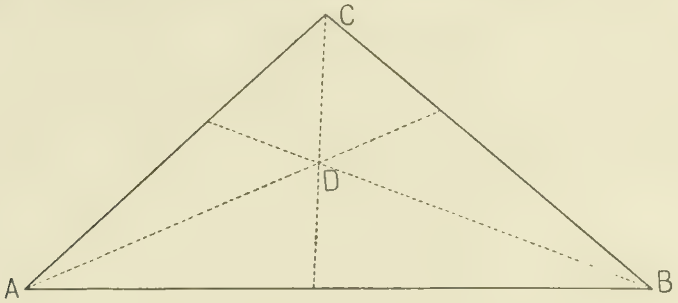

FIG. 10.

The lines AD, BD, and CD, are invariant lines of the transformation T, since they are the joins of two invariant points. The pencil of lines through one of these invariant points, for example A, undergoes a one-dimensional transformation, which leaves three of its lines invariant: such a one-dimensional transformation is an identical transformation and leaves all lines through A invariant (Chapter I, Theorem

4). For the same reason all lines through B and C are invariant lines.

Now the intersection of two invariant lines is an invariant point; hence every point in the plane is an invariant point, for it is the intersection of at least two invariant lines; also all lines are necessarily invariant lines. A transformation T which leaves every point of the plane invariant is an identical transformation.

If the fourth invariant point D be taken on one side of the invariant triangle ABC, for example on BC, then the one-dimensional transformation along the invariant line BC leaves three points B, C, D invariant and therefore leaves all points on the line invariant. Consequently all lines through A are invariant lines, for each has two invariant points, one at A and the other at its intersection with BC. All points on a line through A are not invariant points of the transformation T, which is therefore not an identical transformation. This case will be discussed later.

In the same way it may be shown that a collineation of the plane which leaves four lines invariant, no three of which pass through a point, leaves all lines and all points of the plane invariant and is an identical collineation.

THEOREM 6. **A plane collineation T which leaves invariant four points forming a quadrangle or four lines forming a quadrilateral leaves all points and lines of the plane invariant and is an identical collineation.**

§ 2. Geometric Construction of Plane Collineations.

86. *Geometric Methods.* Thus far in this chapter we have considered the plane collineation, or two-dimensional projective transformation, from the analytic point of view. We shall now reconsider the same subject and obtain the same results by means of geometric construction. Each method is alone sufficient for the foundations of the theory of collineations, but

the best way to obtain a complete mastery of the subject is to approach it from both points of view and then carefully compare the results. The broader outlook thus obtained more than compensates the reader for the extra time and labor expended in learning two methods.

87. *Perspective Projection.* Let two planes π and π' intersect in a line l; from any point P, not in either plane, draw lines to all points of π. Each of the lines is cut by π' in a single point. Two points, A and A' in π and π' respectively, collinear with P are called corresponding points in the two planes. By means of the bundle of rays through P the points of π are projected into the points of π'. Every plane through P cuts π and π' in a pair of corresponding lines that meet on l. It should be observed that to the line g, joining two points A and B in π, corresponds the line g' joining A' and B', their corresponding points in π'. Also to the point A, the intersection of a pair of lines g and h in π, corresponds A', the point of intersection of their corresponding lines g' and h' in π'.

This method of constructing corresponding points and lines in π and π' is called a *perspective projection* of π on π'. By means of this perspective projection whose vertex is at P, we establish a one-to-one correspondence of the points and lines of the two planes π and π'. This one-to-one correspondence is not without exceptions; but these exceptions may be removed by means of special assumptions.

88. *Line at Infinity.* The plane through P parallel to π' cuts π in a line j; the plane through P parallel to π cuts π' in a line i. If we assume with Euclid that through a given point P one and only one plane can be passed parallel to a given plane, and if we further assume that two parallel planes intersect in an infinitely distant line, then our one-to-one correspondence of points and lines is without exception. The line j in π corresponds to the *line at infinity* in π', and the line at infinity in π corresponds to the line i in π'. Thus by introducing the hypothesis that all infinitely distant points in a plane lie on a *line at infinity* the one-to-one correspond-

ence established by perspective projection is made perfectly general; compare Chapter I, article 46.

89. *Perspective Ranges and Pencils.* Let p be a plane through P cutting l in O, and π and π' in a and a' respectively. The ranges of points on a and a' are in perspective position and projectively related as defined in article 51; their point of intersection O is a self-corresponding point of the two ranges.

The pencil of planes intersecting in the line PO is cut by π and π' in two pencils of lines which have a one-to-one correspondence, corresponding lines of the two pencils being coplanar with PO. Their common line l is a self-corresponding line of the two pencils. Two pencils of rays which are the sections of a pencil of planes by two other planes are said to be in perspective position and are projectively related.

90. *Self-corresponding Points and Lines.* The aggregate of all points in a plane is called a field of points and the aggregate of all lines is called a field of lines. When two fields of points are connected by a perspective projection all points common to the two fields, *i. e.*, all points on the line l, are self-corresponding points of the two fields. When two fields of lines are connected by a perspective projection the common line l is a self-corresponding line of the two fields. The line l and all points on it are the only self-corresponding elements of the two fields.

91. *Invariance of Cross-ratios.* The cross-ratios of any four collinear points in π and of their four corresponding points in π' are equal. This follows from the fact that if four points in π lie on a line, say a, their four corresponding points in π' lie on a' which meets a in O, a point on l. The ranges on a and a' are projectively related and in perspective position, and the invariance of cross-ratios of corresponding points was proved in art. 48, Chap. I.

In like manner the cross-ratio of any four concurrent lines in π is equal to that of their four corresponding lines in π', for both sets of lines are cut by l in the same set of four points.

THEOREM 7. Perspective projection of one plane upon another establishes a one-to-one correspondence between the points and also between the lines of the two planes. The line of intersection of the two planes is a self-corresponding line and every point on it a self-corresponding point; the cross-ratio of any four collinear points or concurrent lines in one plane is equal to the cross-ratio of their four corresponding points or lines in the other plane.

92. *Non-perspective Projection.* In Chapter I, § 6, we found two methods of projecting a range of points on a line into a range on another line, viz.: perspective projection and non-perspective projection. Two ranges rendered projective to one another by either of these methods were found to have precisely the same properties, differing only in the fact that two perspective ranges have a self-corresponding point, while two non-perspective ranges do not. Projectivity was found to be one and the same property in each case, the difference being only a result of position. Perspective projection was shown to be only a special case of non-perspective projection.

We have thus far in the present section defined perspective projection of one plane upon another and have investigated the properties of two fields of points and lines connected by a perspective projection. We wish now to consider something analogous to the non-perspective projection of Chapter I, article 49. Let us take two fields of points π and π' related by a perspective projection and, while the correspondence remains unaltered, shift one of the planes, say π, into a new position so that π and π' will intersect in a new line l'. The lines joining corresponding points will no longer meet in a point P and there will be no self-corresponding points of the two fields. We wish to find a method of constructing the point A' in π' corresponding to a given point A in π, and the line a' in π' corresponding to a given line a in π. Such a method, if found, might well be called a *non-perspective projection* of π on π'. Judging by analogy we should expect to find perspective projection appearing as a special case of such a non-perspective projection.

93. *Two Corresponding Conics, K and K′.* Take as before two planes π and π', intersecting in a line l. In the plane π draw any conic K touching l at L; and in π' another conic K' also touching l but at another point, L'. From P, Fig. 11, any point in π not on K, draw two tangents to K; these will intersect l in two points, Q and R. From Q and R

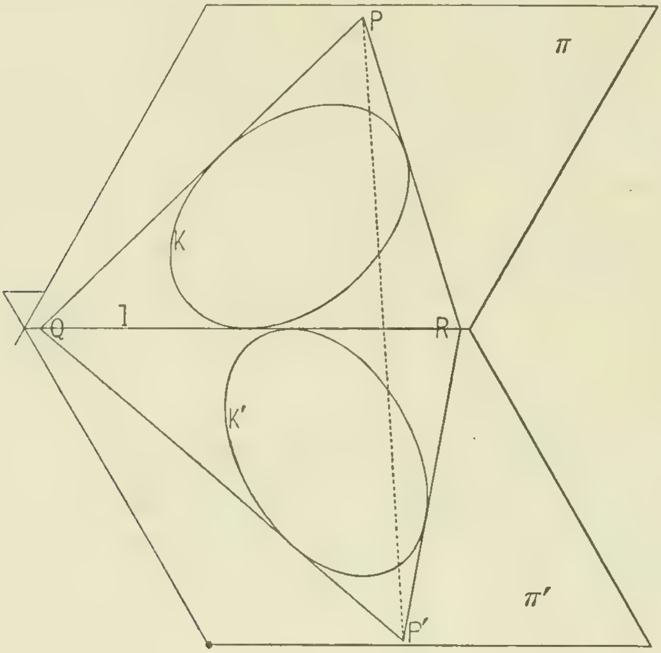

FIG. 11.

draw tangents to K' in π'; these will intersect in a point P'. P and P' in π and π' respectively are called a pair of corresponding points of the two planes. It is evident that this construction determines a one-to-one correspondence between the points not on K and K' respectively of the two planes π and π'.

Let P_1 be any point on the line PQ; it is evident from the construction that its corresponding point P_1' will lie on $P'Q$.

Thus to the points of the range on a tangent to K correspond the points of the range on a tangent to K', corresponding tangents to K and K' meeting on l. Hence the construction transforms the tangents to K into the tangents to K' and determines a one-to-one correspondence between the two sets of tangents.

Suppose that the point P moves along PQ and approaches A, its point of tangency with K; the construction shows that the corresponding point P' will approach A' the point of tangency of $P'Q$ and K'. Hence in the limit the point of tangency of $P'Q$ with K' corresponds to the point of tangency of PQ with K. Therefore the construction determines a one-to-one correspondence between the points of K and the points of K'; the tangents at corresponding points meeting on l.

Let P and P_1 be two points in π not on K, and let the tangents drawn from them to K meet l in Q and R, Q_1 and R_1, respectively. Let t be any tangent meeting these four fixed tangents in A, B, A_1, B_1. Since t and l are both tangents to K, the cross-ratios (ABA_1B_1) and (QRQ_1R_1) are equal. The construction transforms P and P_1 into P' and P_1'; the tangents from these points to K into the tangents from P' and P_1' to K'; the tangent t into the tangent t' and the points A, B, A_1, B_1 on t into A', B', A_1', B_1', the points where t' cuts the tangents from P' and P_1' to K'. Since t' and l are tangents to K', the cross-ratios $(A'B'A_1'B_1')$ and (QRQ_1R_1) are equal. Hence we have $(ABA_1B_1) = (A'B'A_1'B_1')$. Therefore the range of points on a tangent to K and the corresponding range of points on the corresponding tangent to K' are projectively related.

We wish to show next that straight lines in π not tangent to K are transformed by our construction into straight lines in π'. Let P_1 be a point in π not on one of the tangents from P to K and let a pencil of lines be drawn in π through P_1 cutting PQ and PR in two ranges of points, which are therefore projectively related and in perspective position. Our construction transforms this configuration in π into the following configuration in π'; the range of points on PQ goes over into a projectively related range on $P'Q$; also the range on PR goes over into a projectively related range on $P'R$. Since the ranges on PQ and PR are projectively related, the corresponding ranges on $P'Q$ and $P'R$ are also projectively related. These ranges are also in perspective position, since P, the self-corresponding point of the perspective ranges on PQ and PR, goes over into P', which is therefore a self-corresponding point of the ranges on $P'Q$ and $P'R$. Since these ranges have a self-corresponding point, it follows that the lines joining their corresponding points meet in a point. This point is determined by the intersection of any two of the lines joining corresponding points. We know that in π the two tangents from P_1 to K cut PQ and PR in pairs of corresponding points. These tangents go over into the two tangents in π' fromP_1' to K' and hence these tangents also join pairs of corresponding points on $P'Q$ and $P'R$. Therefore, P_1' is the vertex of the pencil in π' projecting the range on $P'Q$ into that on $P'R$. Let X and Y be a pair of corresponding points on PQ and PR respectively. The points P_1, X, Y in π are collinear; their corresponding points in π' are P_1', X', Y', which are also collinear, therefore our construction transforms straight lines in π into straight lines in π'.

Since our construction transforms collinear ranges on tangents to K into collinear ranges on tangents to K', and every collinear range in π into a collinear range in π', it readily follows that any range of points on a line g in π is transformed into a projectively related range on g' in π'. Also any pencil of rays through a point P in π is transformed into a projectively related pencil through P' in π'.

We have now proved that our construction by means of two conics K and K' correlates the plane π to π' in precisely the same manner as the perspective projection described in art. 87, except in the matter of self-corresponding points and lines; *i. e.*, it establishes a one-to-one correspondence between the points of the two planes, between the lines of the two planes; and the cross-ratio of any four collinear points, or any four concurrent lines, in the one plane is equal to the cross-ratio of the four corresponding points or lines in the other plane. Our construction is therefore fully entitled to be called a *non-perspective* projection of π into π'.

THEOREM 8. Two conics K and K' in the planes π and π' respectively, both touching their line of intersection l, determine a non-perspective projection of π into π'.

94. *Real Non-perspective Projection.* In the special case that the conics K and K' are real conics, the non-perspective projection transforms real points and lines into real points and lines. In this case the points inside of K cannot be constructed directly. It is evident that the polar of P with respect to K projects into the polar of P' with respect to K'. Let P be a point inside of K and let p be its polar with respect to K. Choose two points on p outside of K' and construct the corresponding points in π'. These points determine the line p', the projection of p. Construct the pole of p' with respect to K'; this point is P', the projection of P.

95. *Four Pairs of Corresponding Lines.* Let four lines a, b, c, d, be chosen in π and their four corresponding lines a', b', c', d' in π'. Let each set be so chosen that no three of them are concurrent and no two meet on l. The five lines a, b, c, d, l determine a conic K in π, and the five a', b', c', d', l' determine K' in π'. The two conics K and K' determine uniquely and completely a non-perspective projection of π on π'.

THEOREM 9. Four pairs of corresponding lines in the most general position are necessary and sufficient to determine a non-perspective projection of one field of lines on another.

96. *Four Pairs of Corresponding Points.* Let us choose four points A, B, C, D in π and four points A', B', C', D' in π'. Let these points in each plane be so chosen that no three of them are collinear. Let us assume that A and A', B and B', etc., are pairs of corresponding points in a non-perspective

projection. We wish to show that the projection is thereby uniquely and completely determined.

Connect AB, BC, CD, and DA, by lines which we shall call a, b, c, d, respectively. The lines joining corresponding points are their corresponding lines a', b', c', d', respectively. The lines a, b, c, d, l in π and a', b', c', d', l' in π' determine the conics K and K' respectively; and these conics determine a non-perspective projection of π on π'.

But the four points $ABCD$ determine six lines, and these taken four at a time give us fifteen quadrilaterals. These fifteen quadrilaterals give rise to fifteen different pairs of conics, which determine either fifteen different projections of π on π', or the same projection in fifteen different ways, or more than one and less than fifteen, some of them being duplicated.

Let us consider the lines AB, AC, etc., in π and their corresponding lines $A'B'$, $A'C'$, etc., in π'. The intersections of opposite sides of the quadrangle $A'B'C'D'$ correspond to the intersections of the corresponding opposite sides of the quadrangle $ABCD$. In this way three new pairs of corresponding points are determined. New quadrangles may be formed out of these seven points in each plane, and thus other pairs of corresponding points obtained; and so on indefinitely. Hence when four pairs of corresponding points are given in the two planes, an unlimited number of pairs of corresponding points are determined. These considerations show that four pairs of corresponding points in the two planes determine one and only one non-perspective projection of π on π'. It will be shown later that there are ∞^2 different constructions of the same non-perspective projection.

THEOREM 10. Four pairs of corresponding points in the most general position are necessary and sufficient to determine a non-perspective projection of one field of points on another.

97. *Projective Transformation.* Since the constructions in the two planes are exactly alike, the operations of the last article are strictly reversible. By means of the conics K and

K', the configuration in π' corresponding to any given configuration in π can be constructed. And further, one of the planes as π' may be revolved about l until it coincides with π, and then the constructions all thought of as in the same plane. We shall make constant use of this last conception.

By means of a non-perspective projection a field of points in π' can be constructed corresponding to a given field in π*. By a revolution about l this new field of points may be brought back to π and both fields of points thought of as existing in the same plane. This operation of projecting the points in π into a new system of points in π' and, by revolving about l, bringing the new system back to π will be called a *projective transformation* or collineation of the plane π. Such a projective transformation is determined and completely constructed by means of two conics K and K' in the plane π and touching a line l.

THEOREM 11. A projective transformation or collineation of the points and lines of a plane is completely determined by means of two conics K and K' both touching a fixed line l of the plane.

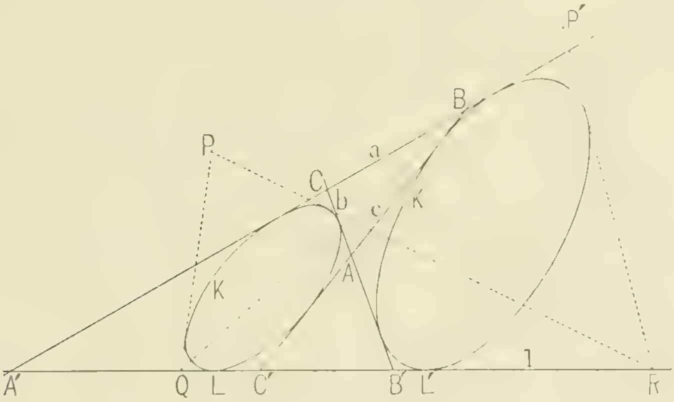

FIG. 12.

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* The lines joining corresponding points of the planes π and π' form a linear congruence (Strahlen-Congruenz) of the third order and first class. See Reye's Geometrie der Lage, II. Band, p. 94.

98. *Invariant Lines and Points.* The two conics K and K' touching the line l will have generally three other common tangents designated by a, b, c, Fig. 12. We shall next examine the relation of these lines to the transformation determined by the two conics K and K'. Let us take a point X on the line c, and draw two tangents from it to the conic K; one of these tangents is the line c which cuts l at C'; the other cuts l at some point as Q'. The corresponding point to X is found by drawing tangents to K' from C' and Q'. One of these tangents is again the line c; the tangent from Q' to K' intersects c in X' the corresponding point to X. In like manner every point on the line c is transformed into a point on the line c; in other words, the line c is an invariant line of the transformation. Similarly the lines a and b are invariant lines of the transformation. The fixed line l is not an invariant line, although a common tangent to the two conics.

The points A, B, C, which are the intersections of the invariant lines a, b, c, are invariant points of the transformation. This is evident from the fact that the two tangents from A to K are the lines b and c; the two tangents from C' and B' to K' are also b and c, which intersect at A, the starting point. Thus A is a self-corresponding point of the transformation; the same is true of B and C.

It is easy to see from the construction that these three lines are the only ones left invariant by the transformation determined by K and K'. In particular cases where the conics K and K' are especially related to one another, *e. g.*, touch one another, the invariant figure may be different. These special cases will be determined later.

THEOREM 12. The three common tangents, other than l, to the two conics K and K' are invariant lines, and their three points of intersection are invariant points of the transformation.

99. *∞^2 Different Constructions of the Same Collineation.* There are ∞^2 conics touching the three invariant lines a, b, c of the collineation T. From these ∞^2 conics may be formed

∞^4 pairs of conics. Among these ∞^4 pairs of conics there are ∞^2 pairs which give the same collineation T. To show this let us choose any conic C touching a, b, and c; the collineation T transforms C into C' touching a, b, c, and some other common tangent m. Corresponding tangents to C and C' are concurrent with m. These conics C and C' and the line m may be used to construct the collineation T in the same way that K, K', and l were used. It is evident that to each of the conics touching a, b, and c there is a corresponding conic; hence there are ∞^2 different constructions of the same collineation.

THEOREM 13. A collineation T can be constructed by means of a pair of conics in ∞^2 different ways.

100. *A Second Construction.* From the self-dualistic character of a plane collineation it is evident that the construction in the plane dualistic to that developed in the last paragraph also holds. This new construction may be deduced from the last by the principle of duality, or it may be developed independently from first principles. We shall take the latter course for the sake of the methods employed, and also for the sake of the wider view of the whole subject thus obtained.

101. *Two Intersecting Conics, K and K_1.* Suppose that a collineation T transforms a point S into S_1 and S_1 into S_2. The pencil of lines through S is transformed into the pencil of lines through S_1. The original pencil through S and the derived pencil through S_1 are projectively related, and hence the locus of the intersection of corresponding rays of the two pencils is a conic K passing through both S and S_1. In like manner the pencil through S_1 is transformed into the pencil through S_2 and the locus of the intersection of the corresponding rays in these two pencils is a second conic, K_1, passing through both S_1 and S_2, Fig. 13. Since the pencils

through S and S_1 are transformed into the pencils through S_1 and S_2, respectively, K must be transformed into K_1.

The line SS_1 is a common ray of the two pencils through S and S_1; considered as a ray of the pencil through S it is transformed into the tangent to K at S_1. Since S is transformed into S_1 and S_1 into S_2, the line SS_1 is transformed into the line S_1S_2. Hence S_2 is the point where the tangent to K at S_1 cuts K_1.

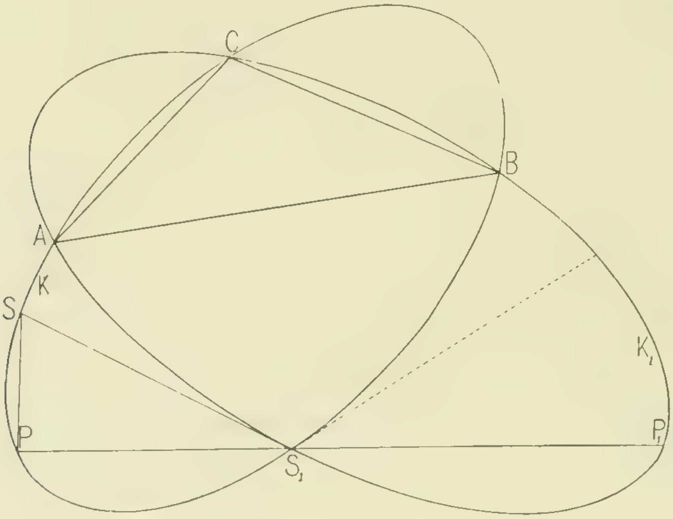

FIG. 13.

Consider a point P on K and its corresponding point P_1 on K_1. The lines SP and S_1P, since they meet on K, are corresponding rays of the two pencils through S and S_1. But S is transformed into S_1 and P into P_1; hence SP and S_1P_1 are corresponding lines of the same two pencils. Therefore, S_1P and S_1P_1 are the same straight line since they both correspond to SP. Hence P and P_1, corresponding points on the two conics K and K_1, are collinear with S_1. Hence we have the important result:*

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*Reye-Holgate, *Geometry of Position*, p. 137.

THEOREM 14. The collineation T, which transforms S into S_1, and S_1 into S_2, transforms the conic K, determined by the projective pencils through S and S_1, into the conic K_1, determined by the projective pencils through S_1 and S_2; every pair of corresponding points on K and K_1 are collinear with S_1.

102. *Construction of a Collineation by Means of K and K_1.* By making use of the principle that corresponding points on K and K_1 are collinear with S_1, we can construct the line g_1 which corresponds to g, any line of the plane.

The line g cuts K in two points, Q and R; join Q and R to S_1; these joins cut K_1 in R_1 and S_1, corresponding points to R and S; the line joining R_1 and S_1 is the line g_1 which corresponds to g.

The transformation T transforms a point P into P_1; if P be given, we may find P_1 by drawing any two lines g and g' through P cutting K; find by the above construction the corresponding lines g_1 and g_1'; these intersect in P_1, the point which corresponds to P.

If a line g does not intersect K, the construction of g_1 may be accomplished by choosing two points, G and G', on g and constructing their corresponding points, G_1 and G_1'; these two new points determine g_1.

If g is a tangent to K, g_1 will be a tangent to K_1 and the points of contact will be collinear with S_1. If the given line g passes through S_1 and cuts K in P and K_1 in P_1, the corresponding line is found by joining P_1 and S_2.

THEOREM 15. A collineation of the plane can be constructed by means of two conics K and K_1 intersecting in a point S_1.

103. *Invariant Points and Lines.* The conics K and K_1 intersect in S_1 and generally in three other points A, B, C. Since any line through S_1 cuts K and K_1 in a pair of corresponding points, it follows that A, B, and C are self-corresponding points on K and K_1. In other words, A, B, and C are invariant points of the collineation T. The lines AB, BC, and CA are self-corresponding or invariant lines of the

collineation T. For example, the line AB cuts K in A and B and the corresponding line cuts K_1 in the corresponding points; but these are also A and B; hence the line AB is transformed into itself.

104. *∞^2 Different Constructions of the Same Collineation.* There are ∞^2 conics passing through the three invariant points A, B, C. From these ∞^2 conics one can form ∞^4 pairs of conics. Out of these ∞^4 pairs of conics, ∞^2 pairs give the same collineation. Let us choose any conic L passing through A, B, and C. The collineation T transforms L into L_1 intersecting L in A, B, C and V_1. The course of reasoning used in art. 101 shows that corresponding points on L and L_1 are collinear with V_1. The two conics L and L_1 and the point V_1 may be used to construct the collineation T in the same way that K, K_1 and S_1 were used. It is evident that to each of the ∞^2 conics through A, B, and C there is a corresponding conic, and hence there are ∞^2 different constructions of the same collineation T.

THEOREM 16. The same collineation T can be constructed by means of a pair of intersecting conics in ∞^2 different ways.

105. *Comparison of the Two Constructions.* The two methods developed in this chapter for constructing a collineation are dualistic to one another. In order to render this dualism more apparent we give here the principle properties of both methods in the following form:

THEOREM 17. A collineation T is completely determined and constructed by means of two conics K and K_1 having a common $\begin{Bmatrix}\text{tangent } l\\ \text{point } S_1\end{Bmatrix}$; corresponding $\begin{Bmatrix}\text{tangents to}\\ \text{points on}\end{Bmatrix}$ K and K_1 are $\begin{Bmatrix}\text{concurrent with } l\\ \text{collinear with } S_1\end{Bmatrix}$. The other three common $\begin{Bmatrix}\text{tangents to}\\ \text{points on}\end{Bmatrix}$ K and K_1 are the three $\begin{Bmatrix}\text{sides}\\ \text{vertices}\end{Bmatrix}$ of the invariant triangle of the transformation T.

§ 3. Types of Plane Collineations.

106. We have thus far dealt only with the most general form of plane collineations and have avoided the consideration of special cases. These special cases must now be considered, and they lead us to the fundamental conception of types of collineations. We shall show that there are five distinct types of plane collineations each of which is characterized by its invariant figure. A plane collineation is a self-dualistic transformation in the sense that it is both a point-to-point and a line-to-line transformation, and hence every plane figure invariant under a collineation must be a self-dualistic figure. This necessary condition will often enable us to determine whether any given figure can be the invariant figure of a collineation.

107. *Type I.* It has been shown in articles 98 and 103 that the most general form of a plane collineation leaves invariant a triangle. This figure, consisting of three points and three lines, is a self-dualistic figure. If a collineation leaves invariant more or less than three points and three lines forming a non-degenerate triangle, it ceases to be a collineation of type I.

A collineation T of type I leaves invariant a triangle ABC. All points on the side c of the invariant triangle undergo a one-dimensional projective transformation whose two invariant points are A and B. The same is true of the points on the other two sides a and b. Likewise the pencils of lines through A, B, and C, respectively, undergo one-dimensional projective transformations, and in each pencil there are two invariant lines. Hence it is evident that the properties of a plane collineation of this type depend in an intimate manner on the properties of a one-dimensional transformation of the first type.

108. *Type II.* If two vertices of the invariant triangle of type I coincide, then two sides must also coincide; for the

change is a self-dualistic change and the modified figure must be a self-dualistic figure. This modified figure consists of two invariant points, A and B, and two invariant lines, l and l'. Two of the invariant points are on one of the invariant lines; and two of the invariant lines pass through one of the invariant points (Fig. 14, II). This is the invariant figure of type II. The one-dimensional transformations along the line AB and through the point A are of the first type; those along the line l and through the point B are parabolic.

109. *Type III.* If the two points A and B of the invariant figure of type II coincide while the lines l and l' do not coincide, the resulting figure is not self-dualistic; the same is true if the two lines l and l' coincide but not the points A and B. Neither of the resulting figures is self-dualistic, and hence there are no types of collineations in the plane characterized by these figures. But if A and B coincide and at the same time l and l', the change is self-dualistic, and also the modified figure. The invariant figure (14, III) consists of a single invariant line and a single invariant point on the invariant line. Such a figure is called a *lineal element.* This gives us type III. The one-dimensional transformation along the invariant line is parabolic; so also is that of the pencil through the invariant point.

110. *Type IV.* A collineation of the plane which leaves invariant four points of the plane, no three of which lie on a line, is an identical transformation, and leaves every point of the plane invariant. It may happen, however, that a third invariant point is situated on one of the sides of the invariant triangle of type I. In that case every point on this side is an invariant point, art. 8, and hence every line through the opposite vertex is an invariant line. The resulting figure (14, IV), which consists of all the points on a line l and all the lines through a point A not on the line l, is self-dualistic. This is the invariant figure of a collineation of type IV, which is called a *perspective collineation.* The one-dimensional transformations along all lines through A and in all pencils

with vertices on l are of the first type with two invariant elements.

111. *Type V.* A special case of the last figure is also obtained when we assume a third invariant point on the line AB of the invariant figure of type II; likewise when we assume another invariant point on the invariant line of the lineal element of type III. The resulting figure (14, V) is self-dualistic and is the invariant figure of a collineation of type V, which is called an *Elation.* The one-dimensional transformations along all the invariant lines and in all the invariant pencils are parabolic, having one element invariant.

This completes the list of types of collineations of the plane; for if we modify these invariant figures in all possible ways we get no new self-dualistic figures.

THEOREM 18. There are five types of collineations in the plane; each type is characterized by one of the self-dualistic invariant figures of Fig. 14.

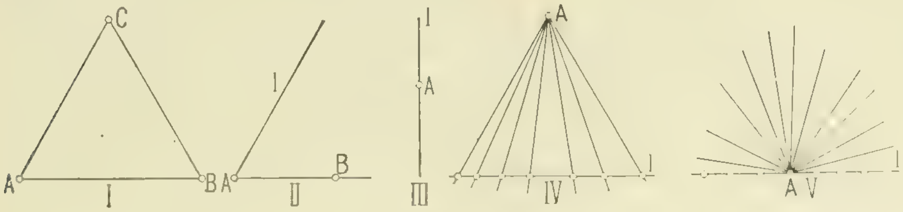

FIG. 14.

112. *Analytic Determination of the Five Types.* Let the collineation be given analytically in the homogeneous form: Thus

$$\begin{aligned}\rho x_1 &= a_1x + b_1y + c_1z,\\ \rho y_1 &= a_2x + b_2y + c_2z,\\ \rho z_1 &= a_3x + b_3y + c_3z.\end{aligned}\qquad(2)$$

Putting $x_1 = x$, $y_1 = y$, and $z_1 = z$ we have three linear equations from which to determine the coordinates of the invariant points. These equations are

$$\begin{aligned}(a_1 - \rho)x + b_1y + c_1z &= 0,\\ a_2x + (b_2 - \rho)y + c_2z &= 0,\\ a_3x + b_3y + (c_3 - \rho)z &= 0.\end{aligned}\qquad(8)$$

When these equations are simultaneous, their resultant vanishes; thus

$$\begin{vmatrix} a_1-\rho & b_1 & c_1 \\ a_2 & b_2-\rho & c_2 \\ a_3 & b_3 & c_3-\rho \end{vmatrix} = 0. \tag{9}$$

This cubic in ρ, designated by $\Delta(\rho)=0$, is called the characteristic equation of the collineation.

There are several cases to be considered. The equation $\Delta(\rho)=0$ may have three single roots ρ_1, ρ_2, ρ_3, one single root ρ_1 and a double root ρ_2, or a triple root ρ_3. Moreover the coefficients of the three equations (2) may satisfy certain conditions so that these three are equivalent to only two equations; or they may satisfy such conditions that the three are equivalent to one. If a root of $\Delta(\rho)=0$, say ρ_1, is substituted for ρ in (8) and no other conditions are imposed on the coefficients, then the three equations (8) are equivalent to only two. They are satisfied by one and only one set of values of the ratios $x:y:z$. (Geometrically speaking the three lines represented by (8) meet in a point.) If the first minors of (9) are all simultaneously zero, then equations (8) are equivalent to only one. They are satisfied by ∞^1 sets of values of the ratios $x:y:z$ and these sets of values satisfy a linear relation. (Geometrically speaking the three lines (8) coincide and may be considered as intersecting at all points of a line.)

113. *Type I.* Let us consider the case where $\Delta(\rho)=0$ has three single roots, ρ_1, ρ_2, ρ_3. In this case the first minors of (9) can not all vanish; for if they do, the conditions for a double root are satisfied. A double root of $\Delta(\rho)=0$ satisfies not only $\Delta(\rho)=0$, but also its derivative $\Delta'(\rho)=0$. But $\Delta'(\rho) = -(\Delta_{11}+\Delta_{22}+\Delta_{33})$, where Δ_{11}, etc., are the first minors of the elements in the principal diagonal of (9). Hence, if the first minors of (9) all vanish, $\Delta(\rho)=0$ has a double root.

If one of these roots, as ρ_1, be substituted for ρ in equations (8), these three equations have a common solution. Solving equations (8) for the ratios $x:y:z$, we thus find the coordinates of an invariant point of the collineation. Substituting successively the three roots of $\Delta(\rho) = 0$ in equations (8) and solving each simultaneous system, we obtain the coordinates of three invariant points.

If equations (2) be interpreted in line coordinates, the same analytic work gives us the coordinates of three invariant lines. These three invariant points and three invariant lines form the vertices and sides of an invariant triangle. Hence, when $\Delta(\rho) = 0$ has three distinct roots, the collineation leaves a triangle invariant and is of type I.

114. *Type II.* If the cubic, $\Delta(\rho) = 0$, has a single root ρ_1 and a double root ρ_2, the invariant figure of the collineation (2) is no longer a triangle. If the first minors of (9) are not all simultaneously zero, the collineation has two invariant points and two invariant lines. The invariant point (or line) corresponding to the double root ρ_2 may be regarded as two coincident invariant points (or lines). The single root ρ_1 gives us an ordinary invariant point (or line). The two invariant lines intersect in one of the invariant points (the double one) and the two invariant points lie on one of the invariant lines (the double one). Hence, when $\Delta(\rho) = 0$ has a double root and the first minors of (9) are not all simultaneously zero, the collineation is of type II.

115. *Type III.* If $\Delta(\rho) = 0$ has a triple root and the first minors of (9) do not all vanish, then there is only one value of ρ that makes (8) simultaneous. Hence the collineation in this case leaves only one point invariant. In line coordinates the same conditions show that the collineation leaves only one line invariant. The condition that the invariant point lies on the invariant line is evidently satisfied, so that the invariant figure is a lineal element. Hence, when $\Delta(\rho) = 0$ has a triple root and the first minors of (9) are not all simultaneously zero, the collineation is of type III.

116. *Type IV.* If $\Delta(\rho) = 0$ has a double root ρ_2 such that when ρ_2 is substituted for ρ in (8) the first minors of (9) are

all simultaneously zero, then the collineation has an invariant figure unlike any of the above cases. The single root ρ_1 gives us a single invariant point (or line). The double root ρ_2 gives ∞^1 invariant points which lie on a line and ∞^1 invariant lines which pass through a point. The single invariant point given by ρ_1 does not lie on the line of invariant points given by ρ_2, since the first minors of (9) do not all vanish when ρ_1 is substituted for ρ in (8). Hence, when the above conditions are satisfied, the invariant figure consists of a line of invariant points and a single invariant point not on this line; the collineation is of type IV.

117. *Type V.* If $\Delta(\rho) = 0$ has a triple root for which all the first minors of (9) are simultaneously zero, there are ∞^1 invariant points which lie on a line and ∞^1 invariant lines which pass through a point. The intersection of these invariant lines is an invariant point which must be one of the points on the line of invariant points. Hence, when the above conditions are satisfied the collineation is of type V.

118. *Geometric Construction of Types of Collineation; Type I.* The method of constructing a collineation given in article 98 also shows in an elegant manner the five types of plane collineations. When the conics K and K' of Fig. 12 are not in contact, the invariant figure is a triangle. Hence, the collineation determined by two conics, which do not touch each other, is of type I.

For special positions of the conics K and K' the invariant figure will be different. Thus special cases arise when the conics touch one another, or touch the line l at the same point, or have contact of the second or third order, etc. These special cases give rise to the other four types of collineations.

119. *Type II.* Let us next consider the case of a collineation where the two conics touch one another. The invariant figure in this case consists of b, the common tangent to the two conics, A their point of contact, and another common tangent a intersecting b at B, Fig. 15a.

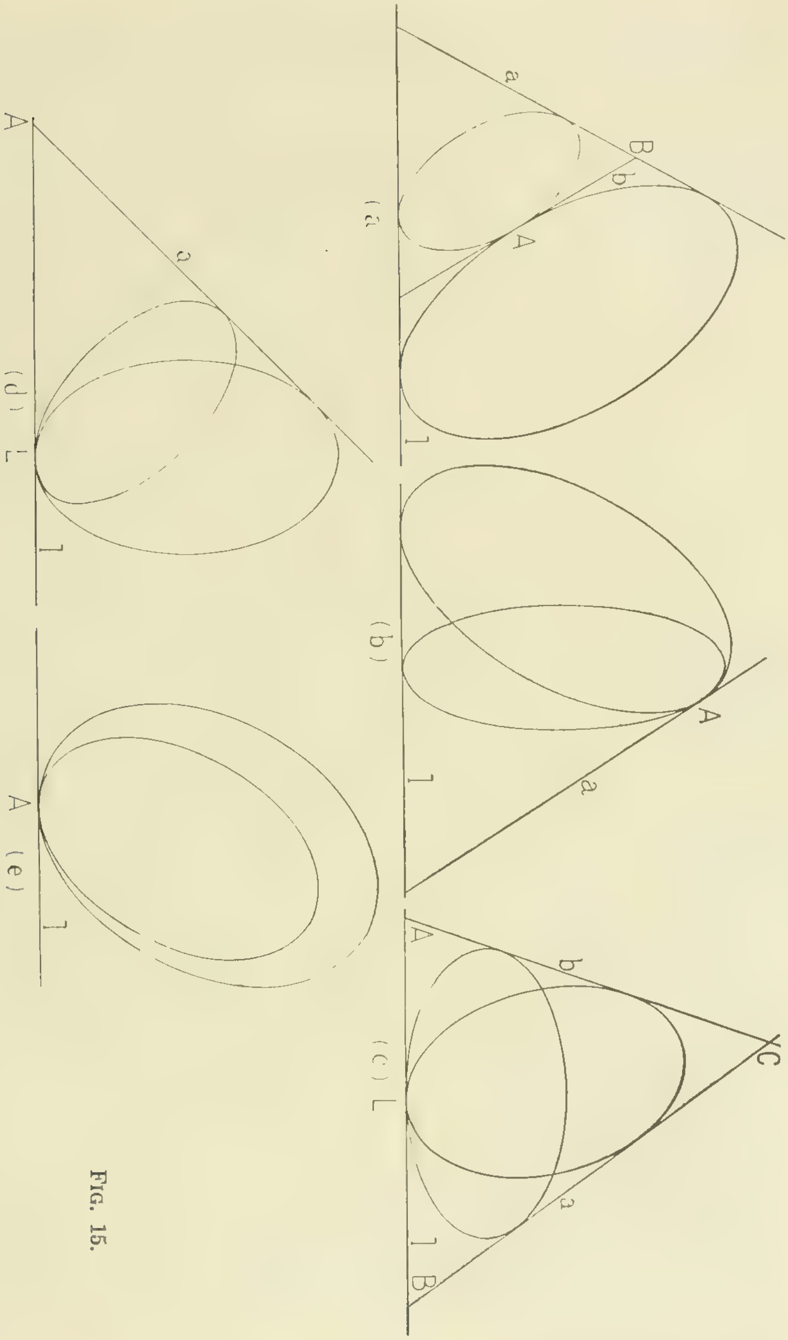
(a)
a
B
b
A
l
(b)
A
a
l
(c)
A
b
C
L
a
l
B
(d)
A
a
L
l
(e)
A
l

FIG. 15.

This kind of a collineation may be considered as a special case of the last, when two sides of the invariant triangle coincide. This case gives us type II.

120. *Type III.* Instead of simple contact, as in the last case, the two conics K and K' may have contact of the second order at a point A. When the conics have a contact of the second order, they have one and only one other common tangent. In this case the common tangent, a, to the two conics at A is the only invariant line of the transformation and the point A is the only invariant point on the invariant line, Fig 15*b*.

In this case the invariant figure consists of a line a and a point A on this line a. This combination of line and point is a lineal element. A collineation of this special kind is of type III and leaves invariant a lineal element.

121. *Type IV.* Again, the two conics may both touch the line l at the same point, the contact being of the first order. In this case the two conics K and K' have two other common tangents, b and a, which intersect at some point C (Fig. 15*c*). It is at once evident from the figure that the transformation determined by K and K' leaves the lines b and a and the point C invariant. A little further consideration of the construction shows that the points A, B, and L on the line l are invariant points. Consider the point δA on b infinitesimally near to A. From δA the two tangents to K are b and a line infinitesimally near to l, meeting l at δL. From δL and A the tangents to K' intersect at $\delta' A$. So that δA is transformed to $\delta' A$. As δA approaches A, $\delta' A$ also approaches A. In the limit A is an invariant point. Similar constructions hold for B and L.

But if a collineation leaves more than two points of a line invariant it leaves all points on the line invariant (art. 8). Therefore, every point on the line l is an invariant point of the transformation. Any line g drawn through C intersects l in some point as G. Therefore, the line g, having two points G and

C invariant, is an invariant line. Thus we see that every line through C is an invariant line. Hence we conclude that the collineation determined by the two conics K and K' touching the line l at the same point leaves the point C, all points of the line l, and all lines through C invariant. If the two remaining points of intersection of K and K' are coincident, *i. e.*, if the two conics have double contact, the resulting collineation is still of the same character and the invariant figure the same. This case is type IV.

122. *Type V.* When the two conics have a contact of the second order at the point L on the line l, the invariant figure takes still another form. In this case only one other common tangent, a, can be drawn to the two conics. This common tangent intersects l at A (Fig. 15*d*). The collineation determined by the two conics in this position leaves invariant all points on the line l and all lines through A. If the two conics have contact of the third order at L, then l is the only common tangent they have (Fig. 15*e*). Such a collineation leaves invariant every point of the line l and every line through L. The invariant figure is the same as before. This constitutes type V.

123. *Perspective Collineations.* Types IV and V constitute what are known as perspective collineations. In article 87 we discussed two projective planes in perspective position. When the plane π' is revolved about the line l until it coincides with π, the resulting collineation in π is called a perspective collineation. Evidently all points on l are self-corresponding or invariant points of the collineation. The perpendicular ray from Q on the plane bisecting the angle between π and π' cuts π and π' in C and C' respectively. Revolution about l brings C' to C, and thus C is an invariant point of the collineation. All lines through C are necessarily invariant lines, for they each pass through two invariant points, viz.: C and an invariant point on l. Therefore, the invariant figure of a perspective collineation is the same as that of type IV. All collineations of type IV are perspective collineations.

124. *Elations.* When the point Q is any point in the plane bisecting the external angle of π and π', the perpendicular from Q on the internal bisecting plane of π and π' meets that plane in a point on l. In this case the invariant point C is a point on l and the invariant pencil of lines through C has its vertex on l. But this is the invariant figure of type V. Such a collineation is called an Elation.* All collineations of type V are elations; elation is a special case of perspective collineation.

125. *Second Construction of the Five Types.* When the two conics K and K_1 of the construction of art. 101 intersect in four points $S_1 ABC$, Fig. 13, we have a collineation of type I. If the two conics have contact of the first order, as for example when A and C coincide, the collineation is of type II and the invariant figure is the degenerate triangle ABl, Fig. 16 (a). If the two conics intersect at S_1 and have contact of the second order at a point A, the invariant figure is a lineal element Al and the collineation is of type III, Fig. 16 (b). If the two conics K and K_1 have contact of the first order at S_1, Fig. 16 (c), then the common tangent to K and K_1 at S_1 is an invariant line, also the lines joining S_1 to the other two points of intersection are invariant lines. Thus we have three invariant lines through S_1 and one invariant line not through S_1. The collineation is of type IV with vertex at S_1 and axis through the other two points of intersection of K and K_1. If the two conics have contact of the second or third order at S_1, Fig. 16 (d) or 16 (e) respectively, the collineation is of type V, as may readily be seen.

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* Lie: Vorlesungen über Continuierliche Gruppen, p. 262.

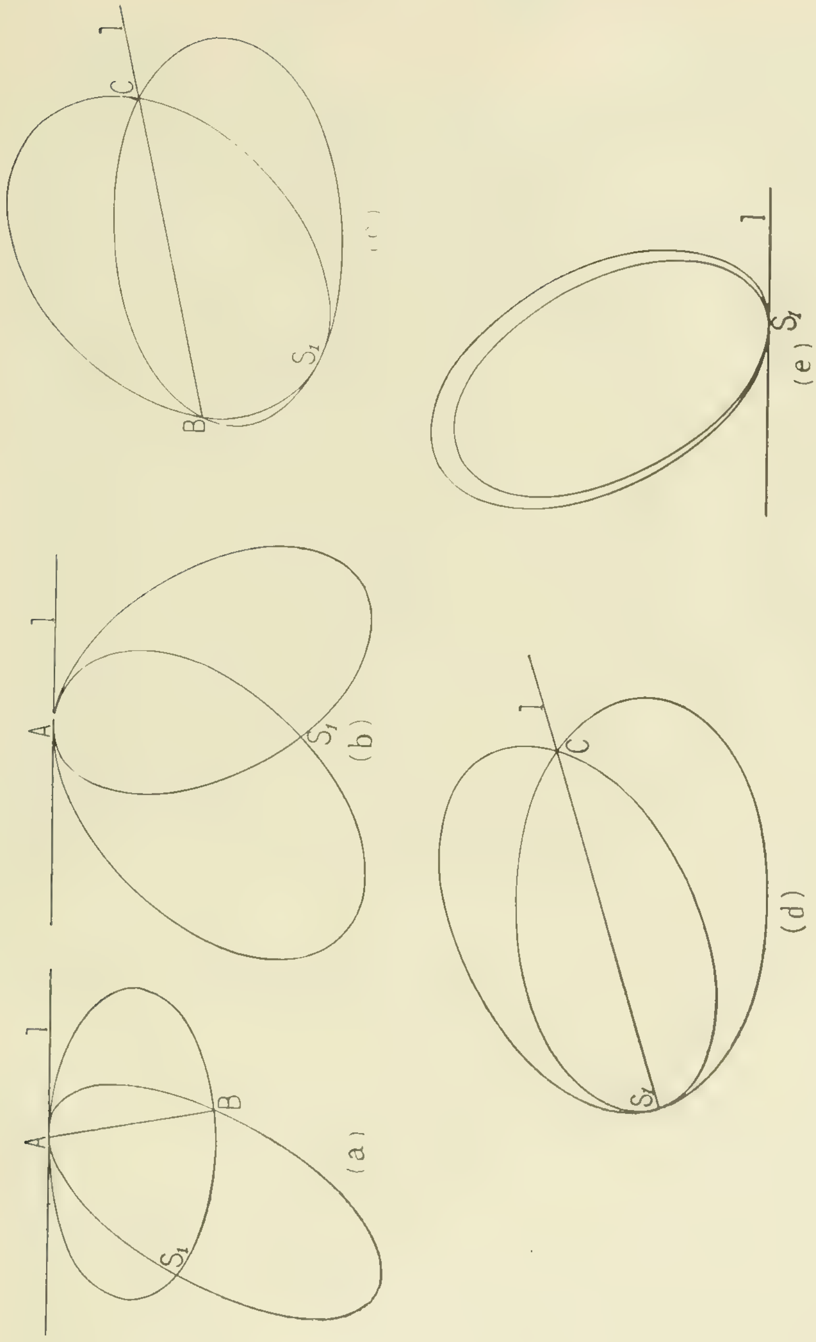
A
l
B
S_1
(a)
A
l
S_1
(b)
B
C
l
S_1
(c)
C
l
S_1
(d)
l
S_1
(e)

FIG. 16.

§ 4. Normal Forms of Equations of the Five Types.

126. In § 2 of chapter I it was shown that there are two distinct types of projective transformations on a line, and that the analytic expressions for these two types can be put into elegant determinant forms in which the constants have definite geometric meanings. It has just been shown that there are five distinct types of plane collineations, and our next task is to find *normal forms* of the equations of these five types. We shall find forms strictly analogous to those already found for one-dimensional transformations, article 17.

We shall first determine the fundamental geometric property of a collineation of type I with reference to its invariant triangle. This geometric property is then expressed in analytic form in terms of the coordinates of a pair of corresponding points and the coordinates of the invariant points of the collineation. We shall first reach an *implicit* normal form and then pass to the *explicit* normal form by solving a set of linear equations. The reduction of the equations of a collineation to their explicit normal form, *i. e.*, the expression of a collineation in terms of its natural parameters by means of an elegant determinant formula, is an analytic result of prime importance.

From the normal form of a collineation of type I we pass readily to the normal forms of collineations of the remaining types.

127. *Three Cross-ratios Whose Product is Unity.* We shall now consider in detail the most general case of a collineation whose invariant figure is a triangle (type I). Let the vertices of the triangle be represented by A, B, C; and the opposite sides by a, b, c, respectively. By means of a collineation T the line a is transformed into itself in such a way that the points B and C on it are invariant points of the transformation. Now we know that the one-dimensional transformation

of the points on a line, which leaves two points of the line invariant, is characterized by the constant cross-ratio of the invariant points and any pair of corresponding points (Art. 15).

Let k_a be the characteristic cross-ratio of the one-dimensional transformation along the line a. In like manner we have transformations of one dimension along each of the invariant lines b and c. We shall call their characteristic cross-ratios k_b and k_c respectively. In reckoning these cross-ratios the points will be taken always in the same order around the triangle. Thus we see that every collineation of type I in the plane determines three characteristic cross-ratios along the three invariant lines. It is also evident that the pencil of lines through the vertex A of the invariant triangle is transformed into itself in such a way that the rays AB and AC are invariant rays of the transformation. Also the cross-ratio of the invariant rays and any pair of corresponding rays of the pencil is constant for all pairs of corresponding rays; this cross-ratio is equal to k_a, the characteristic cross-ratio along the side a opposite A. Similar considerations apply to the pencils of rays through the invariant points B and C. We shall now proceed to show that these three cross-ratios are not independent, but are connected by a very simple relation.

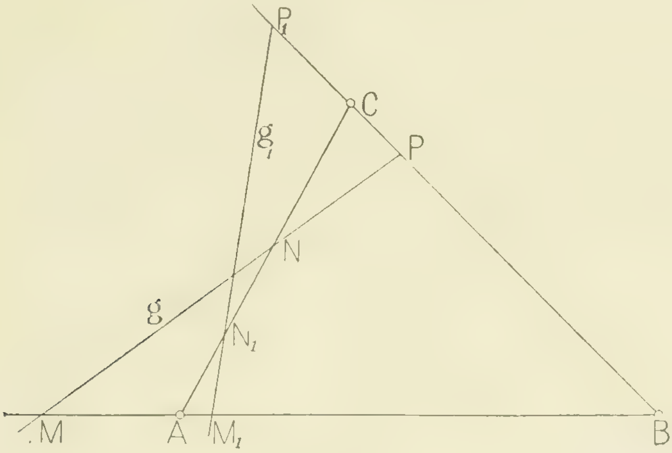

FIG. 17.

Let g and g_1, Fig. 17, be a pair of corresponding lines in the transformation T; let PP_1, NN_1, and MM_1, be the pairs of points of intersection of g and g_1 with the sides of the invariant triangle. Since $k_c = (ABM_1M)$, $k_a = (BCP_1P)$, and $k_b = (CAN_1N)$ (observe the order in which the points are taken) we have

$$k_c k_b k_a = \frac{AM_1 \,.\, BM}{AM \,.\, BM_1} \cdot \frac{BP_1 \,.\, CP}{PB \,.\, CP_1} \cdot \frac{CN_1 \,.\, AN}{CN \,.\, AN_1} \,.$$

But by the theorem of Menelaus* we have

$$\frac{AM .\ BP .\ CN}{BM .\ CP .\ AN} = 1 \quad \text{and} \quad \frac{AM_1 .\ BP_1 .\ CN_1}{BM_1 .\ CP_1 .\ AN_1} = 1.$$

Hence $$k_a k_b k_c = 1 \,.$$

THEOREM 19. Every collineation of type I in the plane determines a characteristic cross-ratio along each of the invariant lines and through each of the invariant points. When these three cross-ratios are reckoned in the same order around the triangle their product is unity.

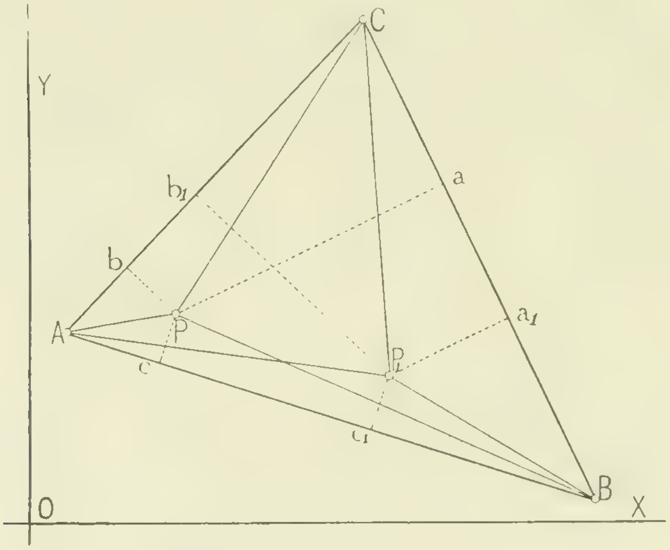

FIG. 18.

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* Cremona, *Elements of Projective Geometry*, page 112.

128. *Cross-ratio of Corresponding Areas.* Let (ABC) be the invariant triangle, Fig. 18, of a collineation T and let P and P_1 be any pair of corresponding points in the plane, P being transformed to P_1; from P and P_1 draw perpendiculars to BC, CA, and AB (dotted lines).

The cross-ratio of the pencil through the vertex C is

$$k_c = C(ABP_1P) = \frac{\sin ACP_1}{\sin BCP_1} : \frac{\sin ACP}{\sin BCP} = \frac{P_1b_1}{P_1a_1} : \frac{Pb}{Pa}.$$

But the perpendiculars from P and P_1 on the sides of the triangle ABC are proportional to the areas of the triangles of which they are the altitudes. Hence

$$k_c = \frac{P_1b_1}{P_1a_1} : \frac{Pb}{Pa} = \frac{\triangle P_1CA}{\triangle P_1BC} : \frac{\triangle PCA}{\triangle PBC}.$$

In like manner we have

$$k_a = \frac{\triangle P_1AB}{\triangle P_1CA} : \frac{\triangle PAB}{\triangle PCA}; \text{ and } k_b = \frac{\triangle P_1BC}{\triangle P_1AB} : \frac{\triangle PBC}{\triangle PAB}.$$

We easily verify that $k_a k_b k_c = 1$.

Since P and P_1 were taken to be a pair of corresponding points in the plane, and since, by Theorem 9, Chapter I, the cross-ratio of the invariant elements and any pair of corresponding elements in a one-dimensional projective transformation is constant for all pairs of corresponding elements, we have found the following important theorem:

THEOREM 20. The cross-ratio of the areas of four triangles whose vertices are any pair of corresponding points in the collineation T and whose bases are any two sides of the invariant triangle of T is constant for all pairs of corresponding points.

129. *Implicit Normal Form of Equations of Type I.* The equations of a collineation are usually given in the form

$$x_1 = \frac{ax + by + c}{a''x + b''y + c''} \text{ and } y_1 = \frac{a'x + b'y + c'}{a''x + b''y + c''}. \qquad (1)$$

When these equations represent a collineation of type I, they can be thrown into a normal form in which the constants in the equation are the coordinates of the three invariant points and the characteristic cross-ratios along the invariant lines.

In Fig. 18 let the coordinates of P be (x, y) and of P_1 be (x_1, y_1); let the coordinates of the three invariant points be (A, B), (A', B'), and (A'', B'').

Expressing the areas of the triangles in the last article in determinant form in terms of the coordinates of their vertices we have

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k_c \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}; \quad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = \frac{1}{k_b} \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}};$$

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k_a \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}.$$

These three forms are not independent and the last may be regarded as superfluous.

Putting $k_c = k$ and $\frac{1}{k_b} = k'$ then the most convenient form is as follows:

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}; \quad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k' \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}. \quad (10)$$

These implicit normal forms are capable of another interpretation; the values of the determinants are proportional to the perpendicular distances from P_1 and P to the sides of the invariant triangle. They express the fact that the cross-ratios of these perpendiculars are constant for all pairs of corresponding points.

130. *Explicit Normal Form of Type I.* Equations (10) are linear in x_1 and y_1, and may be solved for these quantities, giving us the explicit normal form of T. To solve these equations we proceed as follows: Let the equations of the implicit form be written

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k\frac{N}{D}; \qquad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k'\frac{N'}{D}.$$

Expanding and collecting;

$$x_1\left\{D(B-B'')-kN(B'-B'')\right\} - y_1\left\{D(A-A'')-kN(A'-A'')\right\}$$
$$= -D(AB''-A''B)+kN(A'B''-A''B'),$$
$$x_1\left\{D(B-B')-k'N'(B'-B'')\right\} - y_1\left\{D(A-A')-k'N'(A'-A'')\right\}$$
$$= -D(AB'-A'B)+k'N'(A'B''-A''B').$$

Solving by determinants;

$$x_1 = \frac{D\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}(AD-A'kN+A''k'N')}{D\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}(D-kN+k'N')} = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & kA' \\ A'' & B'' & 1 & k'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ A'' & B'' & 1 & k' \end{vmatrix}},$$

$$\tag{11}$$

$$y_1 = \frac{D\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}(BD-B'kN+B''k'N')}{D\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}(D-kN+k'N')} = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & kB' \\ A'' & B'' & 1 & k'B'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ A'' & B'' & 1 & k' \end{vmatrix}}.$$

If we pass from Cartesian to homogeneous coordinates, these forms may be written:

$$\rho x_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}; \quad \rho y_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & B \\ A' & B' & C' & kB' \\ A'' & B'' & C'' & k'B'' \end{vmatrix};$$

$$\rho z_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & C \\ A' & B' & C' & kC' \\ A'' & B'' & C'' & k'C'' \end{vmatrix}. \qquad (12)$$

Making the C's and z's unity in (12) and dividing the first and second by the third, we return to equations (11).

The law of formation of these determinants is evident. The determinant of the invariant triangle is bordered above by x, y, z, on the side by A, kA', $k'A''$, etc.

THEOREM 21. A collineation of type I can be expressed in a symmetric determinant form in which the coefficients are functions only of the eight natural parameters of T.

These explicit normal forms will be of great use to us in the following chapters. By giving to A, B, k, etc., the proper values any assigned collineation of type I can be written down at once. The analogy of these normal forms with the normal forms of type I in one dimension is evident (see Art. 17).

131. *Inverse of T in Normal Form.* If equations (10) be solved for x and y instead of x_1 and y_1, we get the normal form of the inverse of T. From the implicit normal form of T, equations (10), we see that the explicit normal forms of T and its inverse T^{-1} differ only in the fact that k and k' are changed into k^{-1} and k'^{-1}.

132. *Determinant of Normal Form.* The determinant Δ of the normal form of T may be found as follows:

$$\Delta \equiv - \begin{vmatrix} \begin{vmatrix} B & 1 & A \\ B' & 1 & kA' \\ B'' & 1 & k'A'' \end{vmatrix} & \begin{vmatrix} A & 1 & A \\ A' & 1 & kA' \\ A'' & 1 & k'A'' \end{vmatrix} & \begin{vmatrix} A & B & A \\ A' & B' & kA' \\ A'' & B'' & k'A'' \end{vmatrix} \\ \begin{vmatrix} B & 1 & B \\ B' & 1 & kB' \\ B'' & 1 & k'B'' \end{vmatrix} & \begin{vmatrix} A & 1 & B \\ A' & 1 & kB' \\ A'' & 1 & k'B'' \end{vmatrix} & \begin{vmatrix} A & B & B \\ A' & B' & kB' \\ A'' & B'' & k'B'' \end{vmatrix} \\ \begin{vmatrix} B & 1 & 1 \\ B' & 1 & k \\ B'' & 1 & k' \end{vmatrix} & \begin{vmatrix} A & 1 & 1 \\ A' & 1 & k \\ A'' & 1 & k' \end{vmatrix} & \begin{vmatrix} A & B & 1 \\ A' & B' & k \\ A'' & B'' & k' \end{vmatrix} \end{vmatrix}$$

$$\equiv \begin{vmatrix} A & B & 1 \\ kA' & kB' & k \\ k'A'' & k'B'' & k' \end{vmatrix} \times \begin{vmatrix} \begin{vmatrix} B' & 1 \\ B'' & 1 \end{vmatrix} & \begin{vmatrix} A' & 1 \\ A'' & 1 \end{vmatrix} & \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \\ \begin{vmatrix} B & 1 \\ B'' & 1 \end{vmatrix} & \begin{vmatrix} A & 1 \\ A'' & 1 \end{vmatrix} & \begin{vmatrix} A & B \\ A'' & B'' \end{vmatrix} \\ \begin{vmatrix} B & 1 \\ B' & 1 \end{vmatrix} & \begin{vmatrix} A & 1 \\ A' & 1 \end{vmatrix} & \begin{vmatrix} A & B \\ A' & B' \end{vmatrix} \end{vmatrix};$$

$$\equiv k\,k' \begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix} \times \begin{vmatrix} \alpha & \beta & \gamma \\ \alpha' & \beta' & \gamma' \\ \alpha'' & \beta'' & \gamma'' \end{vmatrix};$$

whose α, β', etc., are the minors of A, B', etc., in

$$\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}; \qquad \therefore \Delta \equiv k\,k' \begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}^3.$$

THEOREM 22. The determinant of the normal form of the collineation T is equal to the product of the cross-ratios k and k' into the cube of twice the area of the invariant triangle.

133. *Characteristic Equation of the Normal Form of T.* The characteristic equation of the normal form of T is readily written down as follows, compare Art. 20:

$$\begin{vmatrix} \begin{vmatrix} B & 1 & A \\ B' & 1 & kA' \\ B'' & 1 & k'A'' \end{vmatrix} - \rho & \begin{vmatrix} A & 1 & A \\ A' & 1 & kA' \\ A'' & 1 & k'A'' \end{vmatrix} & \begin{vmatrix} A & B & A \\ A' & B' & kA' \\ A'' & B'' & k'A'' \end{vmatrix} \\ \begin{vmatrix} B & 1 & B \\ B' & 1 & kB' \\ B'' & 1 & k'B'' \end{vmatrix} & \begin{vmatrix} A & 1 & B \\ A' & 1 & kB' \\ A' & 1 & k'B'' \end{vmatrix} - \rho & \begin{vmatrix} A & B & B \\ A' & B' & kB' \\ A'' & B'' & k'B'' \end{vmatrix} \\ \begin{vmatrix} B & 1 & 1 \\ B' & 1 & k \\ B'' & 1 & k' \end{vmatrix} & \begin{vmatrix} A & 1 & 1 \\ A' & 1 & k \\ A'' & 1 & k' \end{vmatrix} & \begin{vmatrix} A & B & 1 \\ A' & B' & k \\ A'' & B'' & k'' \end{vmatrix} - \rho \end{vmatrix} = 0.$$

Expanding this equation we get

$$\rho^3 - (1 + k + k')D\rho^2 + (kk' + k + k')D^2\rho - kk'D^3 = 0, \quad (13)$$

where
$$D = \begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}.$$

This equation factors at once into $(\rho - D)(\rho - kD)(\rho - k'D) = 0$; the three roots are therefore D, kD, $k'D$.

THEOREM 23. The roots of the characteristic equation of the normal form of T are D, kD, $k'D$, where D is twice the area of the invariant triangle of T.

134. *Properties of Type II.* A collineation of type II may be regarded as the limiting case of a collineation of type I when two vertices of the invariant triangle approach coincidence. Let T' be a collineation of type II leaving invariant the figure ABl, Fig. 19. Along the invariant line AB and in the pencil through A there are one-dimensional transformations whose cross-ratios are respectively k and k_p. Along the invariant line l and through the invariant point B there

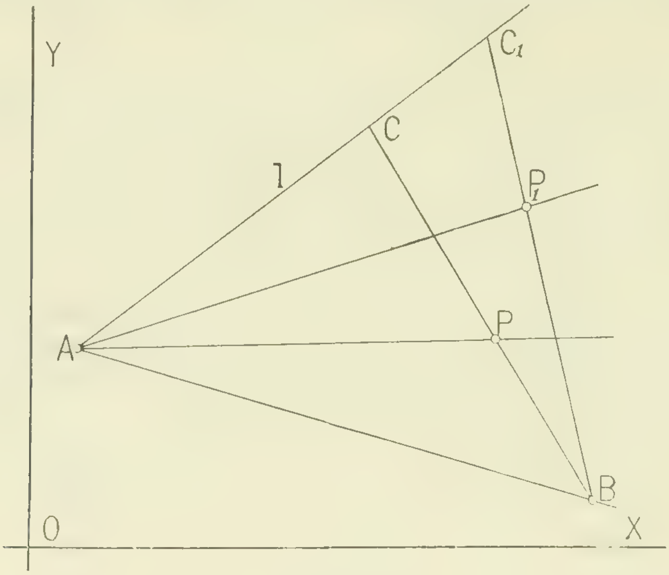

FIG. 19.

are one-dimensional parabolic transformations whose characteristic constants are respectively t and t'. We must first find what relations exist between these constants k and k_p and t and t' respectively.

135. *k and k_p are equal.* In the invariant triangle ABC of type I the product of the cross-ratios k_a, k_b, k_c is unity when they are reckoned in the same order around the triangle. In the triangle let C be moved to coincidence with A, then k_b, along AC, reduces to unity (Art. 19) and we have $k_a k_c = 1$ or $k_c = \frac{1}{k_a}$. Now $k_a = A(CBPP_1)$, hence $\frac{1}{k_a} = A(BCPP_1) = k_c$. Hence the characteristic cross-ratio of the transformation along AB is the same as that of the pencil through A, the order of the elements being as follows: $k = (BAxx_1) = A(l'lPP_1)$, where x and x_1 are a pair of corresponding points in the line AB or l'.

THEOREM 24. **In a collineation of type II the one-dimensional transformations along the invariant line AB and through the invariant point A are both characterized by the same cross-ratio k.**

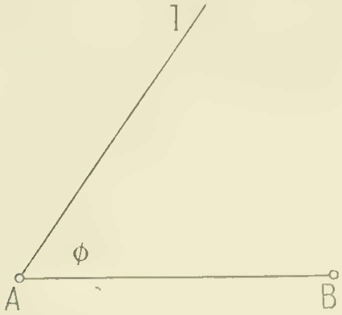

FIG. 19*a*.

136. *Relation between t and t'.* If we consider the figure ABl as the limiting form of the triangle ABC, we can find expressions for the parabolic constants t and t'. Let the angle lAB be denoted by ϕ, Fig. 19*a*. Along the side AC or Al we have $t = \lim. \frac{k'-1}{AC}$ as C approaches A. In the pen-

cil through B we have $t' = \lim\limits_{B=0} \frac{k'-1}{\sin B}$. Hence $\frac{t}{t'} = \lim\limits_{C=A} \frac{\sin B}{AC}$.

But in the triangle ABC, even when C is very near to A, we have $\frac{\sin B}{AC} = \frac{\sin A}{BC}$; and $\lim. \frac{\sin B}{AC} = \lim. \frac{\sin A}{BC} = \frac{\sin \varphi}{AB}$, since AB is the limit of BC.

$$\therefore t = \frac{\sin \varphi}{AB} t'. \tag{14}$$

THEOREM 25. In a collineation of type II the parabolic constants t and t' of the two one-dimensional parabolic transformations along the invariant line Al and through the invariant point B are connected by the relation $t = \frac{\sin \varphi}{AB} t'$.

137. *Normal Form of Type II.* The normal form of the equations of a collineation of type II may be readily obtained from those of type I by considering type II as the limiting form of type I when one of the invariant points, as C, approaches A along the line b. In the normal form of I, equations (11), subtract the second row from the fourth row in each determinant and then divide the fourth row in each determinant by d, the length of the segment AC. Now let C approach A and put $\lim\limits_{A''=A} \frac{A''-A}{d} = c$, $\lim\limits_{A''=A} \frac{B''-B}{d} = c'$, and $\lim\limits_{A''=A} \frac{k'-1}{d} = t$; we then get

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & kA' \\ c & c' & 0 & tA+c \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & t \end{vmatrix}} \qquad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & kB' \\ c & c' & 0 & tB+c' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & t \end{vmatrix}}. \tag{15}$$

Evidently c and c' are cosine and sine of θ the angle which l of the invariant figure of type II makes with the x-axis. If the axes of coordinates are oblique, then c and c' are proportional to the sines of the angles which l makes with the y- and x-axis respectively. In either case c and c' are not independent.

In homogeneous coordinates the above result may be written in the form

$$\rho x_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ cC & c'C & 0 & tA+cC \end{vmatrix}; \quad \rho y_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & B \\ A' & B' & C' & kB' \\ cC & c'C & 0 & tB+c'C \end{vmatrix};$$

$$\rho z_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & C \\ A' & B' & C' & kC' \\ cC & c'C & 0 & tC \end{vmatrix} \qquad (16)$$

The determinant of the normal form (15) of type II, is

$$\Delta = k \begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ c & c' & 0 \end{vmatrix}.$$

THEOREM 26. A collineation of type II is expressible in symmetric determinant form in terms of its seven natural parameters.

138. *Another Method for Type II.*[*] In the last article we derived type II from the general case, type I, by letting A' approach A along the side AA' of the invariant triangle. The following method, which is presented so as to be applicable to type III, also is more general and includes the above method as a special case. A, A' and A'' being the invariant points, draw any continuous curve s from A through A' and A'', Fig. 20. Let the curve have a definite tangent and curvature at A, and let the direction of the tangent and the curvature be continuous throughout s. Now let A' approach A along the curve s, A'' remaining fixed. In the determinants of the normal form of type I substract as above the second row from the third and write $A' - A = \Delta A$, $B' - B = \Delta B$. Divide the new third row through by Δs, the length of the arc AA'.

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[*] The method of this article and its application in the next article is due to Dr. Paul Wernicke. I am under obligations to him for valuable assistance in regard to the normal form of type III, as here presented.

The limits of $\frac{\triangle A}{\triangle s}$, $\frac{\triangle B}{\triangle s}$ and $\frac{k-1}{\triangle s}$ are respectively $\frac{dA}{ds}$, $\frac{dB}{ds}$ and t. The normal form of type II then becomes

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ \frac{dA}{ds} & \frac{dB}{ds} & 0 & At + \frac{dA}{ds} \\ A'' & B'' & 1 & k'A'' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ \frac{dA}{ds} & \frac{dB}{ds} & 0 & t \\ A'' & B'' & 1 & k' \end{vmatrix}}, \quad y_1 = \begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ \frac{dA}{ds} & \frac{dB}{ds} & 0 & Bt + \frac{dB}{ds} \\ A'' & B'' & 1 & k'B'' \end{vmatrix} \;.$$

(Same denominator.)

But $\frac{dA}{ds}$ and $\frac{dB}{ds}$ are the direction cosines of the tangent to s at A, *i. e.*, cosine and sine of its angle with the positive x-axis, and may be replaced by α and β respectively; we have also $\alpha^2 + \beta^2 = 1$.

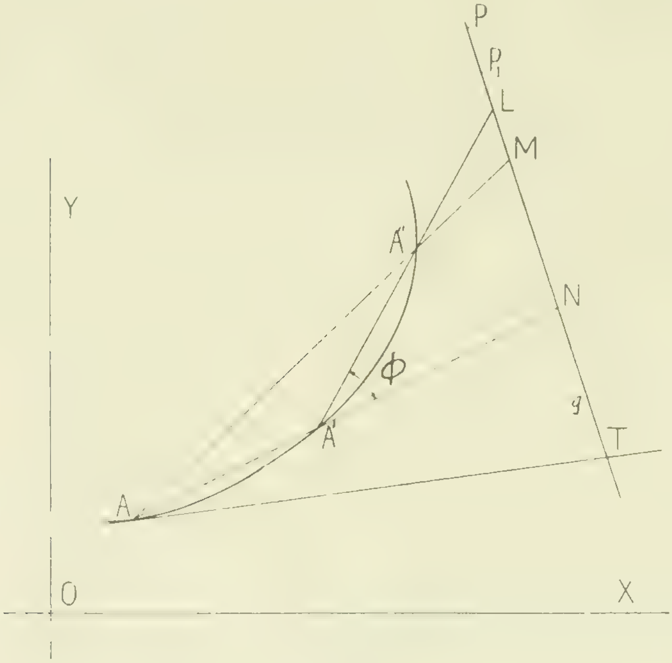

FIG. 20.

139. *Normal Form of Type III.* A collineation of type III is the limiting form of one of type I when all three vertices of the invariant triangle approach coincidence, say at A, and all three sides of the same triangle approach coincidence in the line l. To find the normal form of the equations of type III, we let A'' and A' approach A along the curve s of Fig. 20. When the points A, A', A'' are consecutive points on the curve we may replace the coordinates A', B', A'', B'' by the following expressions: $A' = A + dA$, $B' = B + dB$, $A'' = A + 2dA + d^2A$, $B'' = B + 2dB + d^2B$. We shall divide each determinant of the normal form of type I by twice the area of the invariant triangle $AA'A''$. We may write $2D = AA' . A'A'' \sin\phi$. But when A, A', A'' are consecutive points we have chord $AA' =$ arc $AA' = ds$, chord $A'A'' =$ arc $A'A'' = ds$ and $\sin\phi = \phi = cds$, where c is the curvature of s at A. Hence $2D = cds^3$.

In each determinant of the normal form of type I subtract the second row from the third, subtract twice the third row from the fourth and add the second row to the remainder, divide the new third row through by ds and the new fourth row by cds^2. Substituting the above values of A', etc., we find for the value of x_1

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ \frac{dA}{ds} & \frac{dB}{ds} & 0 & A\frac{(k-1)}{ds} + k\frac{dA}{ds} \\ \frac{d^2A}{cds^2} & \frac{d^2B}{cds^2} & 0 & A\frac{k'-2k+1}{cds^2} + 2\frac{k'-k}{cds}\frac{dA}{ds} + \frac{k'd^2A}{cds^2} \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & \mathrm{B} & 1 & A \\ \frac{dA}{ds} & \frac{dB}{ds} & 0 & \frac{k-1}{ds} \\ \frac{d^2A}{cds^2} & \frac{d^2B}{cds^2} & 0 & \frac{k'-2k+1}{cds^2} \end{vmatrix}},$$

and a similar expression for y_1.

The cross-ratios along the sides of the invariant triangle $AA'A''$ are as follows: k is the cross-ratio along AA', k' is that along AA''; let that along $A'A''$ be k_1, then $k' = kk_1$ (Art. 127). These three k's each approach the limit 1; the limit of $\frac{k-1}{ds}$ is t, as above.

To find the value of $\frac{k'-2k+1}{cds^2}$, we proceed as follows: replace k' by kk_1, and find the values of k and k_1. Let g be the line joining P and P_1, a pair of corresponding points, and let g cut the sides of the invariant triangle in L, M, N, respectively, Fig. 20. Projecting the cross-ratio k from A'' on g we have $k = \frac{LP}{LP_1} \cdot \frac{MP_1}{MP}$. In like manner we have

$$k_1 = \frac{MP}{MP_1} \cdot \frac{NP_1}{NP} = \frac{LP+ML}{LP_1+ML} \cdot \frac{MP_1+NM}{MP+NM},$$

$$= k + \frac{(ML.MP_1+LP.NM+ML.NM) - k\,(ML.MP+NM.LP_1+NM.ML)}{LP_1.MP+ML.MP+LP_1.NM+ML.NM}$$

$$= k + \frac{ML.MP_1+NM.MP-k(ML.MP+NM.MP_1)}{MP_1.NP}.$$

When A, A' and A'' are consecutive points on s, ML and NM are infinitesimals of the same order as AA' and may be made equal to each other. The expression $\frac{k'-2k+1}{cds^2}$ becomes $\frac{k^2-2k+1}{cds^2} + \frac{k}{c}\left(\frac{MP+MP_1}{MP_1.MP}\right)\frac{NM}{ds} \cdot \frac{1-k}{ds} = \frac{t^2}{c} + ht$ where $-h$ is the limit of $\frac{k}{c}\left(\frac{MP+MP_1}{MP_1.NP}\right)\frac{NM}{ds}$. The expression $\frac{k'-k}{cds}$ easily becomes $\frac{t}{c}$.

Since c is the curvature of s at A, $\frac{1}{c} = a$, the radius of curvature of s at A, $\frac{dA}{ds}$ and $\frac{dB}{ds}$ are the direction cosines of the tangent to s at A, and may be replaced by α and β, respectively; also $\alpha^2+\beta^2 = 1$. $a\frac{d^2A}{ds^2}$ and $a\frac{d^2B}{ds^2}$ are the direction cosines of the normal to s at A, and may be replaced by α'

and β', respectively; also $\alpha'^2+\beta'^2=1$, and $\begin{vmatrix}\alpha & \beta\\ \alpha' & \beta'\end{vmatrix}=1$. The normal form of type III may be written

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0\\ A & B & 1 & A\\ \alpha & \beta & 0 & At+\alpha\\ \alpha' & \beta' & 0 & A(at^2+ht)+2at\alpha+\alpha'\end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0\\ A & B & 1 & 1\\ \alpha & \beta & 0 & t\\ \alpha' & \beta' & 0 & at^2+ht\end{vmatrix}};\tag{17}$$

$$y_1 = \frac{\begin{vmatrix} x & y & 1 & 0\\ A & B & 1 & B\\ \alpha & \beta & 0 & Bt+\beta\\ \alpha' & \beta' & 0 & B(at^2+ht)+2at\beta+\beta'\end{vmatrix}}{\text{(Same denominator.)}}.$$

140. *Properties of Type IV.* A collineation S of type IV leaves invariant a point O and every point on l, a line not passing through O, and the pencil of rays through O, Fig. 21.

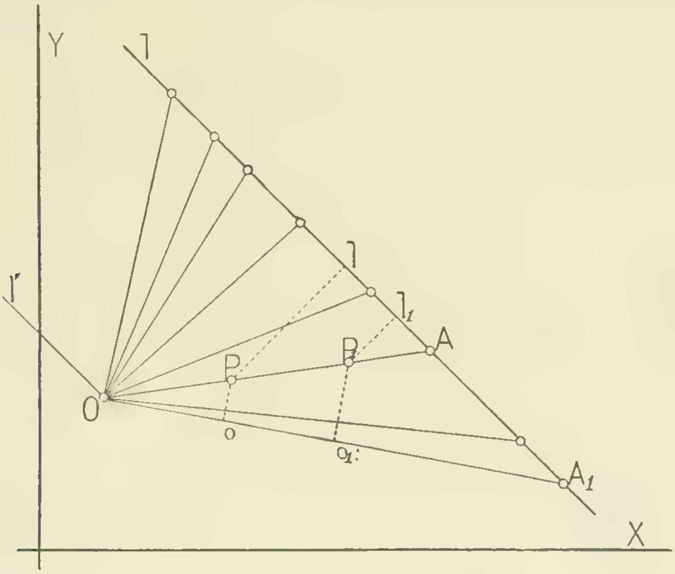

FIG. 21.

In the special case that the point O lies on l, the collineation of type IV reduces to one of type V. Collineations of these two types are perspective collineations. Fixing our attention on the fundamental invariant figure we see that every invariant line in the plane except l has on it two invariant points, O and its point of intersection with l; also every pencil having its vertex A on l is an invariant pencil in the plane, and has in it two invariant rays, l and the line AO. The effect of a perspective collineation of type IV is to move a point P along the invariant line OPP_1 to P_1. Thus we have on each of the invariant lines through O a one-dimensional transformation with two invariant points O and A. Likewise in each of the invariant pencils with vertex on l we have a one-dimensional transformation of the same kind. A one-dimensional transformation is characterized by the constant cross-ratio of the invariant elements and every pair of corresponding elements. Thus along the line AO we have $(AOPP_1) = k$. Let A_1 be another point of l; in the pencil with vertex at A_1 we have the cross-ratio $A_1(AOPP_1) = k$. Since every line through O cuts this pencil in a range having the same cross-ratio k, it follows that the one-dimensional transformations on all lines through O are characterized by the same constant k; also the one-dimensional transformations in all pencils with vertices on l are characterized by the same value of k.

THEOREM 27. **A perspective collineation S of type IV is completely characterized by its fundamental invariant figure and a characteristic cross-ratio k. The one-dimensional transformations along all invariant lines except l and in all invariant pencils except O of S are characterized by the same cross-ratio k.**

141. *Type IV a Special Case of Type I.* A perspective collineation S of type IV may be regarded as a special case of type I. We proved for type I that $k_a k_b k_c = 1$, where these quantities are the characteristic cross-ratios taken in the same

order around the triangle. Along one side, *e. g.* BC, of the invariant triangle we have $k_a = (CBXX_1)$. If $k_a = 1$ and B and C do not coincide, then X and X_1 must coincide and every point on the line BC is an invariant point, and every line through A is an invariant line of the collineation. Thus when one of the cross-ratios as k_a of type I becomes unity without B and C coinciding, T degenerates into S, a transformation of type IV.

142. *Cross-ratio the Same on all Lines Through A.* Since $k_a = 1$, we have $k_c = \frac{1}{k_b}$; thus the characteristic cross-ratio along CA is the reciprocal of that along AB. Interchanging C and A in the formula for k_b we get the reciprocal of k_b; hence the cross-ratio along AC reckoned from A to C is equal to that along AB reckoned from A to B, *i. e.*, $(CAYY_1) = (BAZZ_1) = k_c$. The cross-ratio of the pencil through C is $k_c = C(BAPP_1)$. But every line through A is now an invariant line and all the lines through A cut the pencil through C in the same cross-ratio k_c. Thus we see again that the collineation S produces one-dimensional transformations along each of the invariant lines through A and these one-dimensional transformations are all characterized by the same cross-ratio k.

143. *Type IV a Special Case of Type II.* A transformation T' of type II is characterized by a loxodromic one-dimensional transformation along its invariant line AB and a parabolic one-dimensional transformation along its invariant line Al. If l, the characteristic constant of the parabolic transformation along Al, be equal to zero, the transformation along Al is the identical transformation and every point on Al is an invariant point. The invariant figure is now the point B, all lines through B, and all points on l. For T' it was proved that the characteristic cross-ratios along AB and through

A were equal. Hence it follows in the degenerate form S that the characteristic cross-ratio along any invariant line is equal to that in any invariant pencil.

THEOREM 28. A collineation of type I degenerates into one of type IV whenever the cross-ratio along any side of the invariant triangle is unity and the two vertices on that side do not coincide. A collineation of type II degenerates into one of type IV when the parabolic constant along Al is zero.

144. *Normal Form of Type IV.* The equations of the normal form of type IV are gotten from those of type I by making $k' = k$, $k = 1$ or $k' = 1$. In the first case the line joining (A', B') and (A'', B'') is the line of invariant points. If $k' = 1$, the line of invariant points is that joining (A, B) and (A'', B''); if $k = 1$, the line of invariant points is the line joining (A, B) and (A', B').

The same equations may also be gotten by making $t = 0$ in the normal form of type II. The line Al of Fig. 19 then becomes the line of invariant points and B is the isolated invariant point. The last equations are:

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & kA' \\ c & c' & 0 & c \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & 0 \end{vmatrix}}, \quad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & kB' \\ c & c' & 0 & c' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & 0 \end{vmatrix}}. \qquad (18)$$

145. *Properties of Type V.* In the case of an elation the invariant figure consists of all points on a line l and all lines through a point O on l. An elation S' transforms a point P of the plane into P_1 some other point on the line OP. On each of the invariant lines through O there is a one-dimensional parabolic transformation having its single invariant point at O. Every pencil of lines having its vertex at A, any point on l, is an invariant pencil of the collineation. On each

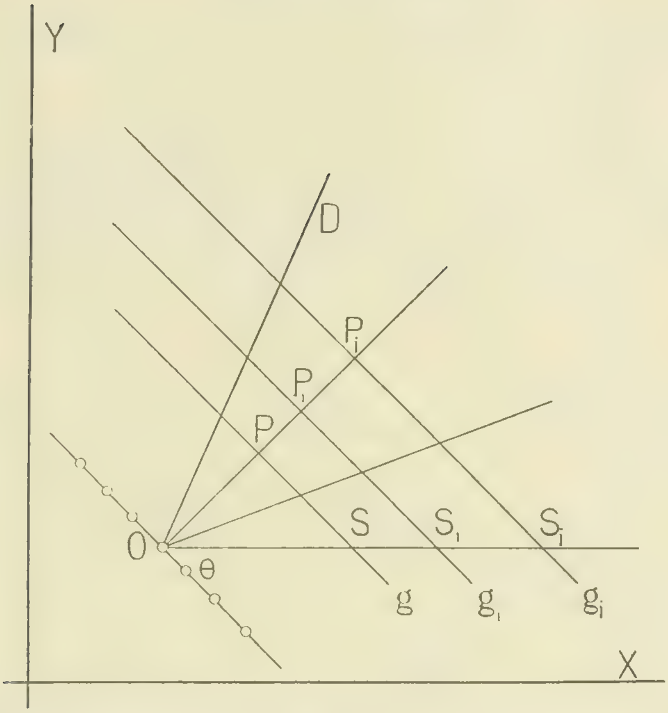

FIG. 22.

of these invariant pencils there is a one-dimensional parabolic transformation having l for its single invariant line.

Let S' be an elation leaving invariant the fundamental figure of Fig. 22. A line g parallel to l will be transformed into g_1 also parallel to l; for g and g_1 both belong to a pencil whose vertex is the point at affinity on l. The line at infinity will be transformed by S' into some line as g_i. Let us first consider here the parabolic transformation along the line OP perpendicular to l; we have, Art. 60,

$$\frac{1}{OP_1} - \frac{1}{OP} = \frac{1}{OP_i} - \frac{1}{\infty} = \frac{1}{OP_i} = t\,.$$

The characteristic constant t is the reciprocal of the segment OP_i where P_i is the point into which the point at infinity is transformed. Along any other line through O as OS making an angle θ with l we have

$$\frac{1}{OS_1}-\frac{1}{OS}=\frac{1}{OS_i}=t\sin\theta.$$

Let A, any point on l, be the vertex of an invariant pencil of rays and let $AO=d$. The elation S' transforms AP into AP_1 and AP_i into AK perpendicular to l at A. Let the angle $PAO=\phi$, $P_1AO=\phi_1$, $P_iAO=\phi_i$, etc. Along OP we have

$$\frac{1}{OP_1}-\frac{1}{OP}=\frac{1}{OP_i};$$

But $\frac{1}{OP_1}=\frac{\cot\phi_1}{d}$, $\frac{1}{OP}=\frac{\cot\phi}{d}$; $\frac{1}{OP_i}=t$. Substituting these values we have

$$\cot\phi_1-\cot\phi=dt. \tag{19}$$

Thus we have the expression for a one-dimensional parabolic transformation of the pencil of lines through a point on l.

THEOREM 29. An elation is completely determined by its fundamental invariant figure and a single characteristic constant. The parabolic constant of the one-dimensional transformation along any given invariant line l' of the fundamental figure is $t\sin\theta$, where t is the parabolic constant along the line perpendicular to the axis and θ is the angle which l' makes with the axis. The parabolic constant of the one-dimensional transformation in any given invariant pencil of the fundamental figure is dt, where d is the distance along the axis from the vertex O of the elation to the vertex of the given pencil.

146. *Type V a Special Case of Type II and of Type III.* We showed in the last article how type IV might be considered as a special case of type II, when $t=0$ in type II. We shall now show that type V is also a special case of type II. In type II when k, the characteristic cross-ratio of the transformation along AB and through A, is unity, these two one-dimensional transformations are both identical transformations and hence every ray through A is an invariant ray; therefore for $k=1$, the transformation $T'(kt)$ of type II degenerates into S' of type V.

A collineation of type V may also be regarded as a special case of type III when the one-dimensional parabolic transformations along Al and through A become identical.

147. *Normal Form of Type V.*[*] The normal form of type V is found by making $k=1$ in that of type II, or by making $a=0$ and $h=0$ in that of type III. Making $k=1$ and the usual reductions in equations (15) we get

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ c_1 & c'_1 & 0 & c_1 \\ c & c' & 0 & At+c \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ c_1 & c'_1 & 0 & 0 \\ c & c' & 0 & t \end{vmatrix}}, \text{ and } y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ c_1 & c'_1 & 0 & c_1' \\ c & c' & 0 & Bt+c' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ c_1 & c'_1 & 0 & 0 \\ c & c' & 0 & t \end{vmatrix}}, \qquad (20)$$

§5. Canonical Forms of Collineations.

The normal forms of plane collineations given in §4 are perfectly general. The axes of reference have no special relations to the invariant figure of the collineation, but the normal forms of the equations may often be greatly simplified by choosing the axes of reference in special positions with respect to the invariant figure. The equation of a collineation in a very simple form will be designated as a *canonical* form. Often a given collineation can be reduced to two or more very simple forms; in such a case we speak of two or more canonical forms of the same collineation.

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[*] The normal forms of the equations of the five types of plane collineations were first given by Prof. Gabriele Torrelli in the *Rendiconti di Circolo Matematico di Palermo*, Tome viii, pp. 41–54. They were found independently by the writer and published by him in the *Kansas University Quarterly*, vol. viii, pp. 45–66. As presented here they differ in minor details from Torrelli's forms and from my own previous forms.

148. *Canonical Forms of Type I.* Let a collineation T of type I be given in the homogeneous normal form

$$\rho x_1 = \begin{array}{|cccc|} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{array}; \quad \rho y_1 = \begin{array}{|cccc|} x & y & z & 0 \\ A & B & C & B \\ A' & B' & C' & kB' \\ A'' & B'' & C'' & k'B'' \end{array};$$

$$\rho z_1 = \begin{array}{|cccc|} x & y & z & 0 \\ A & B & C & C \\ A' & B' & C' & kC' \\ A'' & B'' & C'' & k'C'' \end{array}. \tag{12}$$

If the invariant triangle be taken as the triangle of reference these equations are greatly reduced. Let the coordinates of the vertices of the invariant triangle be $(0, 0, C)$, $(A', 0, 0)$, $(0, B'', 0)$. Substituting these values in equations (12) we get

$$\rho x_1 = kA'B''Cx, \quad \rho y_1 = k'A'B''Cy, \quad \rho z_1 = A'B''Cz.$$

Setting $\rho = \rho' A'B''C$ where ρ' is a new proportionality factor we have

$$\begin{cases} \rho' x_1 = kx, \\ \rho' y_1 = k'y, \\ \rho' z_1 = z. \end{cases} \tag{21}$$

Equations (21) constitute the homogeneous canonical form of the collineation T, the triangle of reference being the invariant triangle.

We can pass from homogeneous coordinates to Cartesian coordinates by dividing the first and second equations of (12) by the third, and then making the z's and C's all unity. Also we can get the canonical form of the collineation in Cartesian coordinates by making the same changes in equations (21). Thus we get

$$\begin{cases} x_1 = kx, \\ y_1 = k'y; \end{cases} \tag{22}$$

in which the invariant triangle is made up of the coordinate axes and the line at infinity. The constant cross-ratios along the x- and y-axes are respectively k and k'.

THEOREM 30. The homogeneous and Cartesian canonical forms of a collineation of type I are respectively

$$\begin{cases} \rho x_1 = kx, \\ \rho y_1 = k'y, \\ \rho z_1 = z, \end{cases} \text{ and } \begin{cases} x_1 = kx, \\ y_1 = k'y. \end{cases}$$

149. *Canonical Forms of Type II.* The Cartesian normal form of a collineation of type II is

$$x_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & A \\ A' & B' & 1 & kA' \\ c & c' & 0 & tA+c \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & t \end{vmatrix}}, \quad y_1 = \frac{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & B \\ A' & B' & 1 & kB' \\ c & c' & 0 & tB+c' \end{vmatrix}}{\begin{vmatrix} x & y & 1 & 0 \\ A & B & 1 & 1 \\ A' & B' & 1 & k \\ c & c' & 0 & t \end{vmatrix}}. \tag{15}$$

Let the invariant point (A, B) be taken for the origin and let the axes of x and y be respectively the lines AB and l of Fig. 19. Setting $A=0$, $B=0$, $B'=0$, $c=0$ and $c'=1$, equations (15) reduce to

$$x_1 = \frac{kx}{1+\frac{k-1}{A'}x+ty}, \quad y_1 = \frac{y}{1+\frac{k-1}{A'}x+ty}. \tag{23}$$

This is a convenient form of type II when the invariant figure is in the finite part of the plane. If (A', B') be the point at infinity on the x-axis, then $A'=\infty$ in (23). Making this reduction we get

$$x_1 = \frac{kx}{1+ty}, \quad y_1 = \frac{y}{1+ty}. \tag{24}$$

Another and somewhat better canonical form is obtained when the point (A', B') is at infinity on the y-axis and (A, B) at infinity on the x-axis. In homogeneous form (24) may be written

$$\rho x_1 = kx, \ \rho y_1 = y, \ \rho z_1 = z+ty. \tag{25}$$

Changing x to y, y to z and z to x, this becomes

$$\rho y_1 = ky, \ \rho z_1 = z, \ \rho x_1 = x+tz. \tag{26}$$

Passing back to Cartesian coordinates, this becomes

$$x_1 = x+t, \ y_1 = ky. \tag{27}$$

(A, B) is now the point at the extremity of the x-axis and (A', B') at the extremity of the y-axis.

150. *Canonical Forms of Type III.* The invariant figure of a collineation of type III is a line l and a point A on l. We

wish to find the reduced form of the equations of type III when the origin is taken at A and the x-axis along the line l. The normal form of type III is given in equations (17) Art. 139. If the invariant point is at the origin, then $A=0$ and $B=0$. If the invariant line coincides with the axis of x, then $\alpha=1$ and $\beta=0$; for α and β are respectively cosine and sine of the angle which l makes with the axis of x. Since α' and β' are the direction cosines of the normal to s at AB, they become 0 and 1 respectively for this position of the invariant figure. Substituting $A=0$, $B=0$, $\alpha=1$, $\beta=0$, $\alpha'=0$, $\beta'=1$ in (17) we get

$$x_1=\frac{x+2\,at\,y}{1+tx+(at^2+h\,t)\,y}, \quad y_1=\frac{y}{1+tx+(at^2+ht)\,y}. \tag{28}$$

Equations (28) may also be put in the form

$$\begin{aligned} \frac{x_1}{y_1}&=\frac{x}{y}+2at, \\ \frac{1}{y_1}&=\frac{1}{y}+t\,\frac{x}{y}+(at^2+ht). \end{aligned} \tag{28a}$$

Equations (28) constitute the canonical form of type III when the invariant figure is in the finite part of the plane.

A second canonical form is obtained by putting the invariant line at infinity and the invariant point A at the extremity of the y-axis. To do this we make equations (28) homogeneous by introducing z as follows:

$$\begin{aligned} \rho x_1&=x+2\,at\,y, \\ \rho y_1&=y, \\ \rho z_1&=z+tx+(at^2+ht)y. \end{aligned}$$

Changing y into z, z into x, and x into y, these equations become, when z is made unity,

$$x_1=x+ty+(at^2+ht),\ y_1=y+2\,at. \tag{29}$$

THEOREM 31. The canonical forms of a collineation of type III. when the invariant figure is in the finite or infinite part of the plane respectively, are

$$\frac{1}{y_1}=\frac{1}{y}+t\,\frac{x}{y}+at^2+ht, \quad \frac{x_1}{y_1}=\frac{x}{y}+2at;$$

and

$$x_1=x+ty+at^2+ht, \quad y_1=y+2at.$$

151. *Canonical Forms of Type IV.* The canonical forms of a collineation of type IV may be most easily obtained from those of type I or of type II. The homogeneous normal form of type IV, when the invariant figure is in the finite part of the plane, is gotten by making $k=1$, or $k'=1$, or $k'=k$ in equations (21). Thus making $k'=1$, we get

$$\begin{cases} \rho x_1 = kx, \\ \rho y_1 = y, \\ \rho z_1 = z. \end{cases} \tag{30}$$

If we make $k'=1$ in (22) we get the Cartesian canonical form of type IV where the axis of the collineation is the y-axis and the vertex is at infinity on the x-axis. Thus

$$\begin{cases} x_1 = kx, \\ y_1 = y. \end{cases} \tag{31}$$

If $k'=k$ in (22), the collineation is of type IV with the vertex at the origin and the axis of the collineation at infinity. Thus:

$$\begin{cases} x_1 = kx, \\ y_1 = ky. \end{cases} \tag{32}$$

If we make $t=0$ in equations (23), the resulting collineation is of type IV with the axis along the y-axis and the vertex on the x-axis at a distance A' from the origin. We thus get

$$x_1 = \frac{kx}{1+\left(\frac{k-1}{A'}\right)x}, \qquad y_1 = \frac{y}{1+\left(\frac{k-1}{A'}\right)x}. \tag{33}$$

152. *Canonical Forms of Type V.* The canonical forms of type V are readily obtained from those of type II, by making $k=1$; for when $k=1$ in the equations of type II the one-dimensional transformation along the line joining (A, B) and (A', B') is an identical transformation, and type II reduces to type V. Making $k=1$ in (23) we get

$$x_1 = \frac{x}{1+ty}, \quad y_1 = \frac{y}{1+ty}. \tag{34}$$

The vertex is now at the origin and the axis of the elation is the y-axis. Making $k=1$ in (27) we get

$$x_1 = x+t, \ y_1 = y. \tag{35}$$

The axis of the elation is the line at infinity and the vertex is the point at infinity on the x-axis.

Equations (34) and (35) constitute the two essentially different canonical forms of type V, when the vertex is respectively in the finite and infinite part of the plane.

§6. Real Collineations in a Plane.

So far in the present chapter the development of the theory of collineations has proceeded on the assumption that in the defining equations both coefficients and variables are complex numbers. Also in the geometric constructions the points, lines and conics employed were not limited to real elements. We shall now go on to the consideration of the special case of real collineations, *i. e.*, collineations that transform real points and lines into real points and lines. The defining equations of a real collineation are real in both variables and coefficients; the conics used in the construction are always real conics.

153. *Sub-types of Type I.* If the defining equations (1) of a collineation are real, then the cubic equation,

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0, \tag{7}$$

whose roots are the three x-coordinates of the vertices of the invariant triangle, has real coefficients. In the general case when the three roots are distinct, one root is always real and the other two may be either real or conjugate imaginary. We therefore distinguish two sub-types of type I; in case the invariant triangle is real in all of its parts the collineation is said to be *hyperbolic;* in case the invariant triangle has one real and two conjugate imaginary vertices, and hence one real and two conjugate imaginary sides, the collineation is said to be *elliptic.*

154. *The Hyperbolic Case.* The invariant triangle of a hyperbolic collineation is real in all its parts; hence the one-dimensional transformations along the invariant sides and through the invariant points are real and hyperbolic. The cross-ratios k and k' along the lines AB and AC respectively are real numbers. The cross-ratio $\frac{k'}{k}$ through A is also real.

155. *The Elliptic Case.* An elliptic collineation leaves invariant a triangle with one real vertex, A, and two conjugate imaginary vertices, B and C, Fig. 23. The one-dimensional transformation along BC is elliptic. So, also, is that through A. The character of the one-dimensional transformations along the imaginary lines AB and AC cannot be inferred from anything developed in Chapter I. k and k' cannot be two independent complex numbers, for $\frac{k'}{k}$ must be a complex number of the form $e^{i\phi}$. From equations (10) it follows that if (A', B') and (A'', B'') are conjugate imaginary points, then k and k' differ only in the sign of i and hence must be conjugate imaginary numbers.

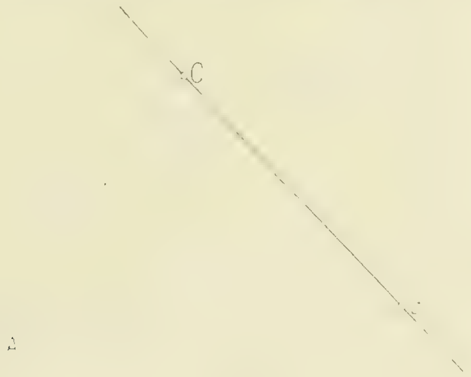

FIG. 23.

THEOREM 32. Real collineations of type I in a plane are either hyperbolic, with three real invariant points, or elliptic with one real and two conjugate imaginary invariant points.

156. *Type II, the Parabolic Case.* When a real hyperbolic collineation of type I degenerates into one of type II by reason of two of its invariant points coinciding, the resulting invariant figure consists of two real points and two real lines. Likewise when a real elliptic collineation degenerates by reason of its two conjugate imaginary points coinciding, the resulting figure again consists of two real points and two real lines. It is clear that a real collineation of type II leaves invariant a figure real in all of its parts. It is also clear that this real collineation of type II stands in the same relation to the hyperbolic and elliptic cases of type I as the real parabolic transformation in one dimension stands to the hyperbolic and elliptic transformations. The one-dimensional transformations along the invariant lines of the invariant figure of type II are a real parabolic and a real hyperbolic transformation, *i. e.*, t and k are always real numbers.

157. *Type III.* The invariant figure of a real collineation of type III consists of one real point and one real line through it. The one-dimensional transformations along the invariant line and through the invariant point are both real parabolic transformations. The constants a, h and t are all real numbers.

158. *Types IV and V.* A real perspective collineation of type IV leaves invariant a figure consisting of all points on a real line BC, and all lines through a real point A. Every invariant line except BC has on it two real invariant points and hence the one-dimensional transformations along the invariant line through A are all real and hyperbolic. Thus the cross-ratio k is always a real number.

The invariant figure of a real collineation of type V consists of all points on a real line l and all lines through a real point A on l. The one-dimensional transformations along the real invariant lines are all real parabolic transformations and t is always a real number.

Exercises in Chapter 2.

A. ANALYTIC THEORY.

1. Obtain equations (3) by solving equations (1) for x and y.

2. Show that the determinant of (3) is the square of the determinant of (1).

3. Discuss the collineation (1) when $\Delta = 0$.

4. Show that the collineation (1) transforms a conic into a conic and a curve of the nth degree into a curve of the nth degree.

5. Find the coordinates of the point which is transformed into the origin by (1); also the coordinates of the point into which the origin is transformed.

6. Find the equation of the line into which the line at infinity is transformed by (1).

7. What values do the coefficients have in (1) when it represents an identical transformation?

8. Give a direct analytic proof that the cross-ratio of four collinear points or four concurrent lines is unaltered by the collineation (1).

9. Prove the theorem of Menelaus quoted in Art. 127.

10. In the normal forms of types I, II and III, show that T and its inverse T^{-1} differ only in this, that k and k' are changed into $1/k$ and $1/k'$, k and t into $1/k$ and $-t$, and t into $-t$, respectively.

11. Show that the determinants of the normal forms of types II, IV, III and V are respectively $k\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ c & c' & 0 \end{vmatrix}$, $k\begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ c & c' & 0 \end{vmatrix}$, 1, and 1.

12. If both terms of the equations (11) of the normal form of T are multiplied (or divided) by any factor $M \neq 0$ or ∞, the determinant of T will be multiplied (or divided) by M.

13. Show that the characteristic equations of the normal forms of types II and III are respectively

$$\rho^3-(2+k)D\rho^2+(2k+1)D^2\rho-kD^3\equiv(\rho-D)^2(\rho-kD)=0,$$

and

$$\rho^3-3\rho^2+3\rho-1\equiv(\rho-1)^3=0.$$

14. Verify the invariance of the points A, A', A'', by substituting their coordinates successively in the normal form of type I.

15. Show directly from the normal form of type I that k and k' are the cross-ratios of the one-dimensional transformations along AA' and AA'' respectively.

16. Solve the problems analogous to 14 and 15 for types II, III, IV and V.

17. Show that k and k' in type I are the ratios of the roots of the characteristic equation of T.

B. GEOMETRIC CONSTRUCTION.

Using the first method of § 2 for constructing a collineation by means of two conics, K and K', touching a line l:

1. Show that the line l in Fig. 12 is transformed into the tangent to K' from the point of contact of K and l.

2. What point corresponds to the point of contact of K and l?

3. Show that the tangent to K from the point of contact of K' and l is transformed into l. What point corresponds to the point of contact of K' and l?

4. Discuss the collineation when both K and K' are parabolas.

5. Show that if the conics K and K' coincide, the collineation T is the identical collineation.

6. Show that interchanging K and K' changes T into its inverse, T^{-1}.

7. Show that a given collineation T can be constructed by

∞^2 different pairs of conics K and K'. One such pair of conics is associated with every line l in the plane.

8. When the line l is the line at infinity in the plane, the two conics K and K' are parabolas; determine their axes.

9. Show that there are ∞^2 collineations of type I leaving the same triangle invariant.

10. Show that there are ∞^2 collineations of type II having the same fundamental invariant figure.

11. Show by this geometric method that there are ∞^8 collineations of the plane.

12. What conditions must the conics satisfy in order that the transformation be a pseudo-transformation.

13. Discuss the cases when K and K' are one or both degenerate conics.

Using the second method for constructing a collineation by means of two conics K and K_1 intersecting in S_1:

14. Show that there are ∞^8 collineations in the plane.

15. Show that every collineation leaving the triangle ABC invariant can be constructed by using the same point S_1 as a vertex.

16. Discuss the method of construction when one or both of the conics K and K_1 are degenerate conics.

17. When K and K_1 are similar and similarly placed conics, the line at infinity is invariant and parallel lines are transformed into parallel lines.

18. When K and K_1 are both circles, show that angles are transformed into equal angles and all figures into similar figures, the ratio of areas being that of K to K_1.

19. When K and K_1 are circles of equal radii, show that the resulting collineation is a rotation of the plane about the one real invariant point through an angle equal to the angle between the radii.

Chapter III.

CONTINUOUS GROUPS OF COLLINEATIONS.

§ 1. Theory of Continuous Groups of Collineations.
§ 2. Resultant of Two Collineations; G_8.
§ 3. Analytic Conditions for a Sub-group of G_8.
§ 4. Groups of Type I Defined by Linear and Quadratic Relations.
§ 5. Groups of Other Types Defined by Linear Relations.
§ 6. Normal Form of Groups of Type I; k-Relations.
§ 7. Fundamental Groups; One-Parameter Groups and Path-Curves.
§ 8. Groups of Perspective Collineations.
§ 9. Groups of Types I, II, and III; Table of Groups.
§ 10. Groups of Real Collineations.
Exercises.

159. The present chapter is devoted to the theory of continuous groups of plane collineations and the determination of all essentially distinct varieties of such groups. We shall also investigate the chief properties of those groups and classify them according to their characteristic properties. In §1 there is developed the fundamental group concept and a general method of handling groups of collineations. In §2 we find in three different ways the resultant of two collineations and establish the existence of the general projective group G_8. The analytic conditions which are necessary and sufficient to define a sub-group of G_8 are developed in §3. In §4 we determine all sub-groups of G_8 for which the defining relations are limited to linear and quadratic relations among the elements of the matrix M of G_8. In §5 we determine all varieties of sub-groups of G_8 that are defined by these same linear and quadratic relations and the additional relations among the elements of M that cause a collineation to degenerate into one of the lower types. The normal form of T is used in §6 to further develop the theory of these groups and to uncover the fundamentally important k-relations. In §7 we investigate by means of the normal form the fundamental

groups of each of the five types of collineation and their one-parameter groups and path-curves; and in §8 the whole theory of groups of perspective collineations is developed by the same instrument. The list of all varieties of continuous sub-groups of G_8 is completed in §9 and a table of these groups is given. Groups of real collineations are treated in §10.

§1. Theory of Continuous Groups of Collineations.

160. *Systems of Collineations.* In the last chapter we studied in detail the properties of each of the five types of plane collineations. We shall now consider the properties of certain infinite systems of these collineations.

Let a collineation T of type I be given in the canonical form, Art. 148,

$$T: \quad \begin{array}{l} x_1 = kx, \\ y_1 = k'y. \end{array} \tag{1}$$

Let k and k' each assume in turn all possible values; we get thereby a system of ∞^2 different collineations. This system of collineations has the important property that each collineation of the system leaves invariant the triangle formed by the x- and y-axes and the line at infinity. The two quantities k and k' are called the parameters of the system, which is therefore called a two-parameter system. In general a system, S_r, of ∞^r collineations $(r < 8)$, which is obtained by varying r independent parameters, is called an r-parameter system. The r-parameter system is said to be a continuous system when it contains the collineations corresponding to every possible complex value of the r independent parameters.

161. *Component and Resultant Collineations.* Let T and T_1 be any two plane collineations. T transforms the points P, P', P'', etc. of the plane into new positions P_1, P_1', P_1'', etc., and the lines l, l', l'', etc. of the plane into new positions l_1, l_1', l_1'', etc. T_1 transforms the points P_1, P_1', P_1'',

etc., and the lines l_1, l_1', l_1'', etc., into P_2, P_2', P_2'', etc., and l_2, l_2', l_2'', respectively. The two collineations T and T_1 acting in succession are equivalent to a single transformation U, which transforms the points P, P', P'', directly to P_2, P_2', P_2'', etc., and the lines l, l', l'', etc., directly to l_2, l_2', l_2'', etc. Since U transforms points into points and lines into lines, it is a collineation* which we may designate by T_2. We say that T_2 is the *resultant* of T and T_1, and that T and T_1 are components of T_2. This relation may be expressed in the form of an equation as follows:

$$T T_1 = T_2,$$

where the two components T and T_1 operate in the order named.

If the two component collineations T and T_1 operate in the reverse order, first T_1 and then T, their resultant T_2' is not always the same as T_2. Thus $T_1 T = T_2'$ and in general $T T_1 \neq T_1 T$. The two resultants T_2 and T_2' are called *conjugate* collineations. When T_2 and T_2' are the same, that is when $T T_1 = T_1 T$, the two collineations T and T_1 are said to be *commutative.*

162. *Groups of Collineations.* A set or system of collineations is called a group, as in Chap. I, Art. 26, when it has the following properties:

First group property. The resultant of any two collineations of the system, taken in either order, is also in the system.

Second group property. The inverse of every collineation in the system is also in the system.

Unless a system of collineations possess both group properties, it is not entitled to be called a group. A system may be so selected that it has the first group property, but not the second. As an example of this we may select the system of one-dimensional transformations given by the equation $x_1 = kx$, where k has all complex values consistent with the condition $|k| =< 1$. This system has the first group property; for $k_2 = kk_1$ and when $|k| < 1$ and $|k_1| < 1$, then also is $|k_2| < 1$ and the first group property is established. The inverse of $x_1 = kx$ is $x = k^{-1}x_1$; when $|k| < 1$, then is $|k^{-1}| > 1$. No transformation in the selected system has its inverse also in the system; hence the system is not a group according to the definition.

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* For a general analytic proof see § 2 of the present chapter.

163. *Parameters of a Group.* When a system of collineations having r parameters is a group, the parameters of the system, as defined in Art. 160, become the parameters of the group, which is called an r-parameter group. For example the system of collineations given by the canonical form of type III, Art. 150,

$$\begin{aligned} x_1 &= x + 2at \\ y_1 &= y + tx + at^2 + ht, \end{aligned} \tag{2}$$

depends upon three parameters, a, h, and t. It will be proved later that this system forms a group. Giving to a, h, and t, all possible values we have ∞^3 collineations which form a three-parameter continuous group; continuous, because of the continuous variation of its parameters. Two consecutive collineations in a continuous group differ only by infinitesimal values of one or more of its parameters.

164. *Classification of Groups.* Continuous groups of collineations may be classified in several ways; according to the number of their parameters, according to the figures which they leave invariant, or according to the types of collineations composing them. The best plan of classification is one that arranges them according to the types of collineations contained in them. We shall therefore speak of groups of type I, type III, etc.; the group mentioned in the previous article is of type III, because the collineations which go to form the group are mostly of type III.

165. *Group Notation.* In chapter II we made use of the notation T, T', T'', S, S', as suitable designations for the five types of collineations. Groups of the five types will be designated respectively by G, G', G'', H, H'. The number

of parameters in the group will be expressed by a subscript; thus a two-parameter group of type I will be written G_2. It is oftentimes desirable to express in the symbol the figure left invariant by all the collineations of the group. This is done by enclosing in parenthesis the symbol for the invariant figure. Thus the symbol $G_3''(Al)$ designates the three-parameter group of type III, leaving invariant the lineal element Al. This is the group whose equations are given in Art. 163.

166. *Groups of the Same Variety.* Two groups composed of the same type of collineations and having the same number and kind of parameters are said to be of the same *variety* when their invariant figures differ only in position, shape or size. For example there are ∞^2 lines in the plane and each line is invariant under the ∞^6 collineations of a six-parameter group. These ∞^2 groups are all of the same variety. It is unnecessary to study more than one group of each variety. We shall enumerate and investigate forty-four varieties of groups of plane collineations. Groups of the same variety are also said to be equivalent, according to the definition of groups given in Art. 35. Thus if G operated on by T gives G' by the formula $G' = T^{-1}GT$, then G and G' are groups of the same variety.

167. *Determination of the Resultant.* A necessary condition that a given system of collineations forms a group is that the system should possess the first group property, *i. e.*, the resultant of any two collineations of the system is one of the same system. The process of finding the resultant of two collineations is one of the most important operations we shall make use of in the present chapter. We must therefore examine the process in detail.

Let a collineation T be given in homogeneous coordinates by the equations,

$$T\colon \begin{aligned} \rho x_1 &= f(x, y, z), \\ \rho y_1 &= \varphi(x, y, z), \\ \rho z_1 &= \quad (x, y, z); \end{aligned} \tag{3}$$

where the three functions f, ϕ and ψ are linear and homogeneous in x, y and z. T transforms the point (x, y, z) into the point (x_1, y_1, z_1). Let T_1 be a second collineation of the same system which transforms the point (x_1, y_1, z_1) into the point (x_2, y_2, z_2). The equations of T_1 are

$$T_1: \begin{aligned} \rho_1 x_2 &= f_1(x_1, y_1, z_1), \\ \rho_1 y_2 &= \phi_1(x_1, y_1, z_1), \\ \rho_1 z_2 &= \psi_1(x_1, y_1, z_1); \end{aligned} \qquad (3')$$

where the functions f_1, ϕ_1 and ψ_1 are of precisely the same form as for T; they differ only in the values of the coefficients. The resultant, T_2, is found by eliminating x_1, y_1, z_1 from the two sets of equations. This gives us a set of equations expressing x_2, y_2, z_2, directly in terms of x, y, z. T_2 may be written

$$T_2: \begin{aligned} \rho_2 x_2 &= f_2(x, y, z), \\ \rho_2 y_2 &= \phi_2(x, y, z), \\ \rho_2 z_2 &= \psi_2(x, y, z). \end{aligned} \qquad (3'')$$

If the functions f_2, ϕ_2, and ψ_2 are of the same form as the corresponding functions for T and T_1 and differ only in the values of the coefficients, then T_2 belongs to the same system as T and T_1 and the system possesses the first group property.

168. *An Illustrative Example.* As a simple illustration let us consider the system of collineations of type II given by equations (26), Art. 149. Let T be given by the equations

$$T: \begin{aligned} \rho x_1 &= x + tz, \\ \rho y_1 &= ky, \\ \rho z_1 &= z. \end{aligned} \qquad (4)$$

Let T_1 be given by the equations

$$T_1: \begin{aligned} \rho_1 x_2 &= x_1 + t_1 z_1, \\ \rho_1 y_2 &= k_1 y_1, \\ \rho_1 z_2 &= z_1. \end{aligned} \qquad (4')$$

Eliminating (x_1, y_1, z_1) from T and T_1 we get T_2 as follows:

$$T_2: \begin{aligned} \rho_2 x_2 &= x + (t + t_1) z, \\ \rho_2 y_2 &= k k_1 y, \\ \rho_2 z_2 &= z, \end{aligned} \qquad (5)$$

which is of the same form as T and T_1.

But T_2 may also be written in the form

$$T_2 : \begin{aligned} \rho_2 x_2 &= x + t_2 z , \\ \rho_2 y_2 &= k_2 y , \\ \rho_2 z_2 &= z . \end{aligned} \tag{4''}$$

Comparing coefficients of corresponding terms in the two forms of T_2 we get

$$\begin{aligned} t_2 &= t + t_1, \\ k_2 &= k k_1. \end{aligned} \tag{6}$$

Since T_2 is of the same form as T and T_1, T_2 belongs to the same system as T and T_1; hence the given system has the first group property.

The inverse of T is given by the equations

$$T^{-1} : \begin{aligned} \rho' x &= x_1 - t z_1 , \\ \rho' y &= k^{-1} y_1 , \\ \rho' z &= z_1 . \end{aligned} \tag{7}$$

Hence the inverse of T is also in the system and the system has the second group property. The system is therefore a group. It has two parameters, k and t, and is thus a two-parameter group. Every collineation in the group leaves invariant the figure (A, A', l); the appropriate symbol of the group is therefore $G'_2 (A A' l)$.

The two equations (6) express the constants of T_2 in terms of those of T and T_1. They are called the equations of condition or *conditional equations* of the group. The number of conditional equations is always just sufficient to determine the constants in T_2.

169. *Subgroups of a Given Group.* Let G_r, $r < 9$, be an r-parameter group of collineations, defined by a set of equations involving r independent parameters. It frequently happens when one or more of these parameters is kept constant and the others are made to vary, that the system of collineations thus selected from G_r has both group properties and is therefore a group within a group, or as it is called a subgroup of the larger group. Subgroups of a given group may often be obtained by setting up a constant relation between two or more of the parameters of an r-parameter group and thus

diminishing the number of independent parameters. Many examples of subgroups obtained by both of the above mentioned methods will be given in the following sections of this chapter.

170. *Invariant Figures and their Groups.* We wish now to prove a theorem of great importance in the determination of continuous groups of collineations. Let T and T_1 be two collineations each of which leaves invariant a certain figure F; since T leaves F invariant and T_1 also leaves F invariant, their resultant T_2 must also leave F invariant. Thus the entire system of collineations leaving a certain figure F invariant has the first group property. Since T leaves F invariant, it transforms points of F into the same or other points of F; hence T^{-1} the inverse of T also transforms the points of F into the points of F, *i. e.*, it leaves F invariant. Thus we see that the system also has the second group property. Such a system is therefore a group. Thus we see that the invariance of a plane figure under a certain system of collineations is a sufficient condition that they form a group. It does not follow that this is a necessary condition.

THEOREM 1. **The system composed of all plane collineations which leave a certain figure invariant forms a group.**

§ 2. Resultant of Two Collineations.

The determination of the resultant of two collineations in the most general form is our immediate problem. When the two component collineations T and T_1 are taken in the most general form and no restrictions laid upon the values of their parameters, their resultant T_2 in either order is assumed to be likewise in the most general form. Any other assumption concerning the form of T_2 is equivalent to some restriction on the values of the parameters of the components. If no restrictions are laid upon the values of the parameters, it is clear that the collineation is of type I. The results of §§

2, 3, and 4 hold only for collineations of type I unless otherwise expressly stated.

171. *Resultant of Two Collineations in Cartesian Form.* Let T and T_1 be given by the Cartesian equations

$$T : x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \quad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}; \tag{8}$$

and

$$T_1 : x_2 = \frac{\alpha_1 x_1 + \beta_1 y_1 + \gamma_1}{\alpha_1 x_1 + \beta_3 y_1 + \gamma_3}, \quad y_2 = \frac{\alpha_2 x_1 + \beta_2 y_1 + \gamma_2}{\alpha_3 x_1 + \beta_3 y_1 + \gamma_3}.$$

The first of these transforms the point (x, y) into (x_1, y_1); the second transforms (x_1, y_1) into (x_2, y_2). The resultant of T and T_1 is a transformation of same kind that transforms (x, y) directly into (x_2, y_2). The equations of this resultant are obtained by eliminating x_1 and y_1 from the equations of T and T_1 They are as follows:

$$T_2 : \begin{cases} x_2 = \dfrac{(a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1)\, x + (b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1)\, y + (c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1)}{(a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3)\, x + (b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3)\, y + (c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3)}, \\[2ex] y_2 = \dfrac{(a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2)\, x + (b_1\alpha_2 + b_2\beta_2 + b_3\gamma_2)\, y + (c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2)}{(a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3)\, x + (b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3)\, y + (c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3)}. \end{cases} \tag{9}$$

The equations are again of the same form as (8) and the transformation T_2 is therefore a collineation.

THEOREM 2. The resultant of two collineations is again a collineation.

172. *Determinant of the Resultant.* The determinant of T_2 is

$$\Delta_2 = \begin{vmatrix} a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 & b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 & c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 \\ a_1\alpha_2 + a_2\beta_2 + a_3\gamma_2 & b_1\alpha_2 + b_2\beta_2 + b_3\gamma_2 & c_1\alpha_2 + c_2\beta_2 + c_3\gamma_2 \\ a_1\alpha_3 + a_2\beta_3 + a_3\gamma_3 & b_1\alpha_3 + b_2\beta_3 + b_3\gamma_3 & c_1\alpha_3 + c_2\beta_3 + c_3\gamma_3 \end{vmatrix} \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} \tag{10}$$

This is equal to the product of the determinants of T and T_1, viz.:

$$\Delta = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \text{ and } \Delta_1 = \begin{vmatrix} \alpha_1 & \beta_1 & \gamma_1 \\ \alpha_2 & \beta_2 & \gamma_2 \\ \alpha_3 & \beta_3 & \gamma_3 \end{vmatrix}.$$

Thus $\Delta_2 = \Delta\,\Delta_1$.

It is evident that the determinant of T_r, the resultant of

$T, T_1, T_2, \ldots T_n$ in any order, is equal to the product of the determinants of the component collineations. Thus

$$\Delta_r = \Delta \Delta_1 \Delta_2 \ldots \Delta_n.$$

THEOREM 3. The determinant of the resultant of two or more collineations is equal to the product of the determinants of the component collineations.

173. *Definition and Notation of a Matrix.* We shall now introduce the notion of a matrix and develop a few of its useful properties. A matrix may be defined as a system of mn quantities arranged in a rectangular array of m rows and n columns. We shall be concerned only with the case where $m = n$ in which case we have a square matrix of order n. It is customary to distinguish a square matrix from a determinant by placing double bars on each side of the array. Thus:

$$M \equiv \left\| \begin{matrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{matrix} \right\|$$

is a square matrix of order 3. The determinant of the matrix is a thing distinct from the matrix itself.

A matrix is said to be of rank r if it contains at least one r-rowed determinant which does not vanish, while all determinants of an order higher than r which the matrix may contain are zero. If the determinant of the matrix does not vanish, the matrix is of rank n.

The coefficients in the equations of a collineation form a matrix which may be taken as the analytic representation of the collineation. We may therefore speak of the matrix M of a collineation T and deal with the matrix instead of the collineation itself.

174. *Multiplication of two Matrices.* The product of two square matrices is defined by the law of composition of two linear transformations as shown in Art. 172. This law of composition expressed in general terms gives the following definition: The product MM_1 of two matrices of the nth order is a matrix of the nth order in which the element that

lies in the ith row and jth column is obtained by multiplying each element of the ith row of M by the corresponding element of jth column of M_1 and adding the results. This law of composition is called Cayley's rule.

It is evident from the definition that the product MM_1 is not in general equal to the product M_1M. Thus the product of two matrices is not definite unless the order of multiplication be specified; or we may say that the multiplication of matrices is not in general commutative, *e. g.*, $MM_1 \neq M_1M$.

Two matrices are said to be conjugate when the rows and columns of one are in the same order the columns and rows of the other. Thus

$$M \equiv \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix} \text{ and } M' \equiv \begin{Vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{Vmatrix}$$

are conjugate matrices.

175. *Determinant Form of the Resultant.* Let T and T_1 be two collineations in homogeneous form as follows:

$$T: \quad \begin{aligned} \rho x_1 &= a_1x + b_1y + c_1z, \\ \rho y_1 &= a_2x + b_2y + c_2z, \\ \rho z_1 &= a_3x + b_3y + c_3z, \end{aligned} \qquad T_1: \quad \begin{aligned} \rho_1 x_2 &= \alpha_1x_1 + \beta_1y_1 + \gamma_1z_1, \\ \rho_1 y_2 &= \alpha_2x_1 + \beta_2y_1 + \gamma_2z_1, \\ \rho_1 z_2 &= \alpha_3x_1 + \beta_3y_1 + \gamma_3z_1. \end{aligned} \tag{11}$$

The collineation T_2 is obtained by eliminating x_1, y_1, z_1 from the above equations. This may be done as follows: Find the inverse of T by solving the three equations of T for x, y, z. Thus we get

$$T^{-1}: \quad \begin{aligned} \frac{\Delta}{\rho} x &= A_1x_1 + A_2y_1 + A_3z_1, \\ \frac{\Delta}{\rho} y &= B_1x_1 + B_2y_1 + B_3z_1, \\ \frac{\Delta}{\rho} z &= C_1x_1 + C_2y_1 + C_3z_1, \end{aligned} \tag{12}$$

where Δ is the determinant of T and A, B, etc., have the same meanings as in equations (3), Chapter II.

The three equations of T^{-1} and the first one of T_1 form a system of four simultaneous linear equations; hence

$$\begin{vmatrix} \frac{\Delta}{\rho}x & A_1 & A_2 & A_3 \\ \frac{\Delta}{\rho}y & B_1 & B_2 & B_3 \\ \frac{\Delta}{\rho}z & C_1 & C_2 & C_3 \\ \rho_1 x_2 & \alpha_1 & \beta_1 & \gamma_1 \end{vmatrix} = 0. \qquad (13)$$

This equation expresses the relation between x, y, z and x_2. Solving this equation for x_2 we get

$$\rho\rho_1 \Delta x_2 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & \alpha_1 \\ A_2 & B_2 & C_2 & \beta_1 \\ A_3 & B_3 & C_3 & \gamma_1 \end{vmatrix}.$$

In like manner we get similar results for y_2 and z_2; thus

$$\rho\rho_1 \Delta y_2 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & \alpha_2 \\ A_2 & B_2 & C_2 & \beta_2 \\ A_3 & B_3 & C_3 & \gamma_2 \end{vmatrix}, \quad \rho\rho_1 \Delta z_2 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & \alpha_3 \\ A_2 & B_2 & C_2 & \beta_3 \\ A_3 & B_3 & C_3 & \gamma_3 \end{vmatrix}. \qquad (14)$$

When the determinants are expanded, Δ divides out of both sides of each equation.

176. *Decomposition of the Normal Form into Factors.* As an illustration of the usefulness of the determinant form of the resultant, we shall make use of it to deduce a theorem of fundamental importance in the theory of collineations. Equations (14) show a striking resemblance to the normal form of type I, Art. 130, and, though the A's, B's, etc., in the two forms have different meanings, they suggest the decomposition of the normal form into factors. Since T_2 expressed in the form of (14) breaks up into factors T and T_1 expressed by (11), so T expressed in the normal form of type I breaks up into factors $T \equiv UV$, where the matrices of U^{-1} and V are as follows:

$$U^{-1} = \begin{Vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{Vmatrix} \quad \text{and } V = \begin{Vmatrix} A & kA' & k'A'' \\ B & kB' & k'B'' \\ C & kC' & k'C'' \end{Vmatrix}.$$

But V factors into $V \equiv \begin{Vmatrix} 1 & 0 & 0 \\ 0 & k & 0 \\ 0 & 0 & k' \end{Vmatrix} \times \begin{Vmatrix} A & A' & A'' \\ B & B' & B'' \\ C & C' & C'' \end{Vmatrix} \equiv T'S.$

Since $U^{-1} = S$, we have $U = S^{-1}$ Hence

$$T = S^{-1}T'S.$$

We see that the collineation S transforms the triangle of reference into the invariant triangle of T; thus the point $(0, 0, 1)$ goes over into (A'', B'', C''), the point $(0, 1, 0)$ into (A', B', C'), and $(1, 0, 0)$ into (A, B, C). Hence S^{-1} transforms the invariant triangle of T into the triangle of reference. The collineation of T is consequently decomposed into three operations acting in the following order: S^{-1} transforms the invariant triangle of T into the triangle of reference, T' leaves the triangle of reference invariant, but transforms every other point in the plane, S transforms the triangle of reference back into the invariant triangle of T.

T' is a collineation in its canonical form, Art. 148, and T is the result of operating on T' by S. T is thus the so-called *transform* of T' by S; T' and T are equivalent collineations.

In like manner the normal forms of each of the other types of collineations may be broken up into the product $S^{-1}T'S$, where S is a collineation depending on the invariant elements of T, and T' is the canonical form of its type.

THEOREM 4. Every collineation T in its normal form may be factored into $S^{-1}T'S$, where T' is the canonical form of its type.

177. *Resultant of T and its Inverse.* Another interesting application of the determinant form of the resultant is its use in finding the resultant of a collineation T and its inverse T^{-1}. Let T be written in the form:

$$T: \quad \begin{aligned} \rho x_1 &= a_1x + b_1y + c_1z, \\ \rho y_1 &= a_2x + b_2y + c_2z, \\ \rho z_1 &= a_3x + b_3y + c_3z, \end{aligned} \tag{11}$$

and let its inverse T^{-1} be written,

$$T^{-1}: \quad \begin{aligned} \frac{\triangle}{\rho}\, x_2 &= A_1x_1 + A_2y_1 + A_3z_1\,, \\ \frac{\triangle}{\rho}\, y_2 &= B_1x_1 + B_2y_1 + B_3z_1\,, \\ \frac{\triangle}{\rho}\, z_2 &= C_1x_1 + C_2y_1 + C_3z_1\,. \end{aligned} \qquad (12)$$

Forming the resultant by the method of Art. 175, we have,

$$T: \quad \Delta^2 x_2 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & A_1 \\ A_2 & B_2 & C_2 & A_2 \\ A_3 & B_3 & C_3 & A_3 \end{vmatrix}, \quad \Delta^2 y_1 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & B_1 \\ A_2 & B_2 & C_2 & B_2 \\ A_3 & B_3 & C_3 & B_3 \end{vmatrix},$$

$$\Delta^2 z_2 = \begin{vmatrix} x & y & z & 0 \\ A_1 & B_1 & C_1 & C_1 \\ A_2 & B_2 & C_2 & C_2 \\ A_3 & B_3 & C_3 & C_3 \end{vmatrix}. \qquad (14)$$

Since $\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix} = \Delta^2$, the equations of T_2 reduce to $x_2 = x_1$, $y_2 = y$, $z_2 = z$; thus T_2 is the identical collineation.

THEOREM 5. The resultant of any collineation T and its inverse is the identical collineation.

178. *Resultant in the Normal Form.* Let T and T_1 be two collineations in homogeneous normal forms,

$$T: \quad \rho x_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}, \quad \rho y_1 = \begin{vmatrix} x & y & z & z \\ A & B & C & B \\ A' & B' & C' & kB' \\ A'' & B'' & C'' & k'B'' \end{vmatrix},$$

$$\rho z_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & C \\ A' & B' & C' & kC' \\ A'' & B'' & C'' & k'C'' \end{vmatrix}. \qquad (15)$$

$$T_1: \quad \rho_1 x_2 = \begin{vmatrix} x_1 & y_1 & z_1 & 0 \\ A_1 & B_1 & C_1 & A_1 \\ A_1' & B_1' & C_1' & k_1A_1' \\ A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix}, \quad \rho_1 y_2 = \begin{vmatrix} x_1 & y_1 & z_1 & 0 \\ A_1 & B_1 & C_1 & B_1 \\ A_1' & B_1' & C_1' & k_1B_1' \\ A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix},$$

$$\rho_1 z_2 = \begin{vmatrix} x_1 & y_1 & z_1 & 0 \\ A_1 & B_1 & C_1 & C_1 \\ A_1' & B_1' & C_1' & k_1C_1' \\ A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}. \qquad (15')$$

Their resultant T_2, is also of the same form:

$$T_2: \quad \rho_2 x_2 = \begin{vmatrix} x & y & z & 0 \\ A_2 & B_2 & C_2 & A_2 \\ A_2' & B_2' & C_2 & k_2A_2' \\ A_2'' & B_2'' & C_2'' & k_2'A_2'' \end{vmatrix}; \quad \rho_2 y_2 = \begin{vmatrix} x & y & z & 0 \\ A_2 & B_2 & C_2 & B_2 \\ A_2' & B_2' & C_2' & k_2B_2' \\ A_2'' & B_2'' & C_2'' & k_2'B_2'' \end{vmatrix};$$

$$\rho_2 z_2 = \begin{vmatrix} x & y & z & 0 \\ A_2 & B_2 & C_2 & C_2 \\ A_2' & B_2' & C_2' & k_2C_2' \\ A_2'' & B_2'' & C_2'' & k_2'C_2'' \end{vmatrix}. \qquad (15'')$$

Eliminating x_1, y_1, and z_1 from the equations of T and T_1, we get T_2 as follows:

$$\rho\rho_1 x_2 = \begin{vmatrix} \triangle x & \triangle y & \triangle z & 0 \\ A_1 & B_1 & C_1 & A_1 \\ A_1' & B_1' & C_1' & k_1A_1' \\ A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix}, \quad \rho\rho_1 y_2 = \begin{vmatrix} \triangle x & \triangle y & \triangle z & 0 \\ A_1 & B_1 & C_1 & B_1 \\ A_1' & B_1' & C_1' & k_1B_1' \\ A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix}$$

$$\rho\rho_1 z_2 = \begin{vmatrix} \triangle x & \triangle y & \triangle z & 0 \\ A_1 & B_1 & C_1 & C_1 \\ A_1' & B_1' & C_1' & k_1C_1' \\ A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}, \qquad (16)$$

where $\triangle x$ is the determinant value of ρx_1, etc.

The resultant may be expressed in the form of determinants of the seventh order, as follows:

$$\rho\rho_1 x_2 = \begin{vmatrix} x & y & z & 0 & 0 & 0 & 0 \\ A & B & C & A & B & C & 0 \\ A' & B' & C' & kA' & kB' & kC' & 0 \\ A'' & B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & 0 & A_1 & B_1 & C_1 & A_1 \\ 0 & 0 & 0 & A_1' & B_1' & C_1' & k_1A_1' \\ 0 & 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix};$$

$$\rho\rho_1 y_2 = \begin{vmatrix} x & y & z & 0 & 0 & 0 & 0 \\ A & B & C & A & B & C & 0 \\ A' & B' & C' & kA' & kB' & kC' & 0 \\ A'' & B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & 0 & A_1 & B_1 & C_1 & B_1 \\ 0 & 0 & 0 & A_1' & B_1' & C_1' & k_1B_1' \\ 0 & 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix};$$

$$\rho\rho_1 z_2 = \begin{vmatrix} x & y & z & 0 & 0 & 0 & 0 \\ A & B & C & A & B & C & 0 \\ A' & B' & C' & kA' & kB' & kC' & 0 \\ A'' & B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & 0 & A_1 & B_1 & C_1 & C_1 \\ 0 & 0 & 0 & A_1' & B_1' & C_1' & k_1C_1' \\ 0 & 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}. \qquad (17)$$

It will be observed that these determinants differ only in the last column.

179. *Equations of Condition.* Equating coefficients of x, y, and z, in the two forms of T_2, viz., equations $(15'')$ and (17), we get the following nine equations:

$$\begin{vmatrix} B_2 & C_2 & A_2 \\ B_2' & C_2' & k_2A_2' \\ B_2'' & C_2'' & k_2'A_2'' \end{vmatrix} = \begin{vmatrix} B & C & A & B & C & 0 \\ B' & C' & kA' & kB' & kC' & 0 \\ B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & A_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1A_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix}. \tag{I}$$

$$\begin{vmatrix} A_2 & C_2 & A_2 \\ A_2' & C_2' & k_2A_2' \\ A_2'' & C_2'' & k_2'A_2'' \end{vmatrix} = \begin{vmatrix} A & C & A & B & C & 0 \\ A' & C' & kA' & kB' & kC' & 0 \\ A'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & A_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1A_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix}. \tag{II}$$

$$\begin{vmatrix} A_2 & B_2 & A_2 \\ A_2' & B_2' & k_2A_2' \\ A_2'' & B_2'' & k_2'A_2'' \end{vmatrix} = \begin{vmatrix} A & B & A & B & C & 0 \\ A' & B' & kA' & kB' & kC' & 0 \\ A'' & B'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & A_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1A_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'A_1'' \end{vmatrix}. \tag{III}$$

$$\begin{vmatrix} B_2 & C_2 & B_2 \\ B_2' & C_2' & k_2B_2' \\ B_2'' & C_2'' & k_2'B_2'' \end{vmatrix} = \begin{vmatrix} B & C & A & B & C & 0 \\ B' & C' & kA' & kB' & kC' & 0 \\ B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & B_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1B_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix}. \tag{IV}$$

$$\begin{vmatrix} A_2 & C_2 & B_2 \\ A_2' & C_2' & k_2B_2' \\ A_2'' & C_2'' & k_2'B_2'' \end{vmatrix} = \begin{vmatrix} A & C & A & B & C & 0 \\ A' & C' & kA' & kB' & kC' & 0 \\ A'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & B_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1B_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix}. \tag{V}$$

$$\begin{vmatrix} A_2 & B_2 & B_2 \\ A_2' & B_2' & k'B_2' \\ A_2'' & B_2'' & k_2'B_2'' \end{vmatrix} = \begin{vmatrix} A & B & A & B & C & 0 \\ A' & B' & kA' & kB' & kC' & 0 \\ A'' & B'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & B_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1B_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'B_1'' \end{vmatrix}. \tag{VI}$$

$$\begin{vmatrix} B_2 & C_2 & C_2 \\ B_2' & C_2' & k_2C_2' \\ B_2'' & C_2'' & k_2'C_2'' \end{vmatrix} = \begin{vmatrix} B & C & A & B & C & 0 \\ B' & C' & kA' & kB' & kC' & 0 \\ B'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & C_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1C_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}. \qquad \text{VII}$$

$$\begin{vmatrix} A_2 & C_2 & C_2 \\ A_2' & C_2' & k_2C_2' \\ A_2'' & C_2'' & k_2'C_2'' \end{vmatrix} = \begin{vmatrix} A & C & A & B & C & 0 \\ A' & C' & kA' & kB' & kC' & 0 \\ A'' & C'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & C_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1C_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}. \qquad \text{VIII}$$

$$\begin{vmatrix} A_2 & B_2 & C_2 \\ A_2' & B_2' & k_2C_2' \\ A_2'' & B_2'' & k_2'C_2'' \end{vmatrix} = \begin{vmatrix} A & B & A & B & C & 0 \\ A' & B' & kA' & kB' & kC' & 0 \\ A'' & B'' & k'A'' & k'B'' & k'C'' & 0 \\ 0 & 0 & A_1 & B_1 & C_1 & C_1 \\ 0 & 0 & A_1' & B_1' & C_1' & k_1C_1' \\ 0 & 0 & A_1'' & B_1'' & C_1'' & k_1'C_1'' \end{vmatrix}. \qquad \text{IX}$$

These nine equations are not independent; dividing each of these equations through by any one of them, we obtain eight independent equations which enable us to express the eight parameters, A_2, B_2, k_2, etc., in terms of A, B, k, A_1, k_1, etc.

179½. *Determinant of the Resultant.* It was shown in Art. 172, that, if T and T_1 are two collineations whose determinants are respectively Δ and Δ_1, the determinant Δ_2 of T_2, the resultant of T and T_1, is equal to the product of Δ and Δ_1; $\Delta_2 = \Delta\Delta$. Making use of the value of the determinant of T in Art. 132 we have the equation

$$k_2k_2'\begin{vmatrix} A_2 & B_2 & C_2 \\ A_2' & B_2' & C_2' \\ A_2'' & B_2'' & C_2'' \end{vmatrix}^3 = k\,k'\,k_1k_1'\begin{vmatrix} A & B & C \\ A' & B' & C' \\ A'' & B'' & C'' \end{vmatrix}^3\begin{vmatrix} A_1 & B_1 & C_1 \\ A_1' & B_1' & C_1' \\ A_1'' & B_1'' & C_1'' \end{vmatrix}^3. \qquad \text{X}$$

180. *The Group G_8.* We have shown in three different ways by the eliminations in Arts. 171, 175 and 178 that the resultant of two collineations is again a collineation. The inverse of a collineation is also a collineation as was shown in Art. 77, and again by the form of T^{-1} in equations (12). Since the system of all the ∞^8 collineations of the plane has both group properties, it is proved that they form a group G_8.

This is called the general collineation group or general projective group of the plane.

THEOREM 6. The system of ∞^8 collineations of the plane form a group G_8.

§ Analytic Conditions for a Subgroup of G_8.

181. We shall develop in this section some fundamental theorems in the theory of groups of collineations and shall establish the necessary and sufficient conditions for the existence of a subgroup of G_8.

We shall use throughout this and the next two sections the homogeneous form of a collineation T and shall take the proportionality factor ρ equal to unity. Thus

$$T: \quad \begin{aligned} x_1 &= a_1x + b_1y + c_1z, \\ y_1 &= a_2x + b_2y + c_2z, \\ z_1 &= a_3x + b_3y + c_3z. \end{aligned} \qquad (11')$$

This form is just as general as that used in Art. 175, and for our purpose far more convenient. In this form three definite numbers x, y, z, are transformed by T into three other definite numbers x_1, y_1, z_1. But if the three equations (11′) be each multiplied by $\rho \neq 0$, we see that T transforms $\rho x, \rho y, \rho z$ into $\rho x_1, \rho y_1, \rho z_1$; *i. e.*, the ratios $x : y : z$ are transformed into $x_1 : y_1 : z_1$.

There are two homogeneous forms of linear transformation each of which is equivalent to the same Cartesian form, viz.: T as above and T; thus

$$T: \quad \begin{aligned} -x_1 &= a_1x + b_1y + c_1z, \\ -y_1 &= a_2x + b_2y + c_2z, \\ -z_1 &= a_3x + b_3y + c_3z. \end{aligned} \qquad (11'')$$

But the ∞^8 transformations T do not form a group. The resultant of two of them is a transformation of the kind T. We shall use only the first kind to represent a collineation and care must be taken in any equations that involve the squaring of the equation T; for such an operation merely introduces the transformations of the second kind. In particular we note that the determinant of T is the negative of the determinant of T; thus $\overline{T}$ may be written

$$\begin{aligned} x_1 &= -a_1 x \quad b_1 y \quad c_1 z, \\ y_1 &= -b_2 x - b_2 y \quad c_2 z, \\ z_1 &= -a_3 x - b_3 y \quad c_3 z. \end{aligned}$$

Whence $\overline{\Delta} = -\Delta$, where Δ is the determinant of T.

182. *Existence of Subgroups of G_8*. Our first question is to ask if the general projective group G_8 contains continuous subgroups.

Let the matrix of a collineation T be

$$M \equiv \begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{Vmatrix};$$

let 'the determinant of the matrix be not zero; and let the characteristic equation of the matrix have three distinct roots, so that the collineation is of the most general type. If all the elements of the matrix vary independently we have the eight-parameter group G_8 of all collineations in the plane. In order to select out from this group a system of collineations with a smaller number of parameters we must reduce the number of independent parameters in the group G_8, *i. e.*, we must impose upon the elements of M one or more relations. This necessary condition is also sufficient; for if we impose r relations upon the elements of M, we reduce by r the number of independent parameters. If this system of collineations is to have the first group property, these relations on the elements of M must be of such a form that if they are imposed upon the elements of the matrices of T and T_1, they will also be satisfied by the elements of the matrix of T_2, where $T_2 = T\,T_1$. It thus appears that a necessary condition for the existence of a subgroup of G_8 is the existence of a set of one or more relations among the elements of M having the property that they are satisfied simultaneously by the elements of the matrices of T, T_1 and T_2.

The existence of subgroups of G_8 is evident geometrically. It was shown in Theorem 1 of this chapter that the invariance of a geometric figure under all the transformations of a certain system is a sufficient condition that they form a group. This may be expressed analytically by saying that the invariance under T of one or more functions of the variables is a sufficient condition for a subgroup of G_8. If a certain function of the variables remains invariant under a linear transformation T, then the corresponding parameters of the original and of the transformed functions must be equal. This implies that one or more relations exist among the elements of the matrix M, and it also gives us a hint as to the form of such relations. Thus it appears that the existence of a proper set of relations among the elements of M is a sufficient condition for a subgroup of G_8.

183. *The Substitutions U and V.* We must first examine the form assumed by the substitution for the elements of M_2 in terms of those of M and M_1. We note first the form assumed by the substitution for the elements of the first row of M_2. From the two forms of M_2, equation (10), we have the substitution,

$$U_1 : \quad \begin{aligned} A_1 &= a_1\alpha_1 + a_2\beta_1 + a_3\gamma_1 , \\ B_1 &= b_1\alpha_1 + b_2\beta_1 + b_3\gamma_1 , \\ C_1 &= c_1\alpha_1 + c_2\beta_1 + c_3\gamma_1 ; \end{aligned} \qquad (18)$$

which is in the form of a linear transformation, the variables being the elements of the first row of M_1, and the matrix of the transformation being the conjugate of M. We also note that the substitutions for the second and third rows of M_2 give us the same linear transformation so far as the matrix is concerned, the variables being the elements of the second and third rows of M_1 respectively. The elements of the three rows of M_1 form therefore three sets of cogredient variables. The three substitutions may be expressed by one formula, thus:

$$U_i : \quad \begin{aligned} A_i &= a_1\alpha_i + a_2\beta_i + a_3\gamma_i , \\ B_i &= b_1\alpha_i + b_2\beta_i + b_3\gamma_i , \\ C_i &= c_1\alpha_i + c_2\beta_i + c_3\gamma_i ; \end{aligned} \qquad (i=1,2,3) \qquad (18')$$

In like manner we see that the substitutions for the elements in the columns of M_2 are in the form of linear transformations in the elements of the corresponding columns of M, the matrix of the transformation being M_1. The transformation on the first column is written thus:

$$V_1 : \quad \begin{aligned} A_1 &= \alpha_1 a_1 + \beta_1 a_2 + \gamma_1 a_3 , \\ A_2 &= \alpha_2 a_1 + \beta_2 a_2 + \gamma_2 a_3 , \\ A_3 &= \alpha_2 a_1 + \beta_3 a_2 + \gamma_3 a_3 ; \end{aligned} \qquad (19)$$

and the others similarly. Hence the following theorem:

THEOREM 7. The substitutions for the elements of M_2 in terms of those of M and M_1 are, in one of two ways, in the form of the same linear transformation in three sets of three variables each. The three sets of cogredient variables are the elements of the three rows of M_1 (or columns of M) while the matrix of the transformation is the conjugate of M (or M_1 itself).

184. *The Set of Relations R.* Our problem is now to determine the properties of a set of relations on the elements of M such that their existence is both a necessary and a sufficient condition for a subgroup of G_s. Let us assume the existence of a system of collineations within G_s having the first group property and hence a set of relations that are all satisfied by the elements of M, M_1, and M_2. Let the set R be r in number and let them be represented by $F_k(a_1, b_1, \ldots ; l_1, l_2, \ldots) = c_k$, $(k = 1, 2, \ldots r)$, where $l_1, l_2 \ldots l_j$ are constants or parameters of the functions F_k. It is conceivable that a given function F of the set may contain all, a part, or none of the parameters of the set; also that one or more of the F's may contain all, a part, or none of the elements of M. Let us substitute for A_1, B_1, etc., in these relations on M_2 their values from equation (10) in terms of the elements of M and M_1. Consider the new relations thus formed as functions of α_1, β_1, etc., the elements of M_1. Since these new relations on α_1, β_1, etc., are by hypothesis of exactly the same form as the original relations on A_1, B_1, etc., they may be written $F_k(\alpha_1, \beta_1, \ldots ; l_1', l_2', \ldots l_j') = c_k$, and we can equate the corresponding parameters of the two sets. We thus get $l_1' = l_1$, $l_2' = l_2$, etc.,

where l_1', l_2', etc., are functions only of l_1, l_2, etc., and the elements of M and may be written $\phi_j(a_1, b_1, \ldots; l_1, l_2, \ldots) = l_j$. This gives us a set of relations among the elements of M. But this set of relations thus obtained among the elements of M must be, at least, a part of the set of assumed relations $F_k(a_1, b_1, \ldots; l_1, l_2, \ldots) = c_k$. We cannot in this way get more relations on the elements of M than our assumed set, for by assumption all the relations among the elements of M are also satisfied by the elements of M_1 and M_2. We may get by this process exactly the assumed set of relations or a part of them. Evidently there is one and only one function ϕ for each parameter l in the set R; and among the constants c_k are the j parameters l_j. The parts played by M and M_1 may be interchanged in the above process.

Let us select one equation $\phi_a = l_a$ of the set and investigate the parameters contained in the function ϕ_a. The function ϕ_a may contain all of the j parameters or only a part of them.

First let us suppose that ϕ_a contains all of the j parameters. Then transforming ϕ_a by U and equating parameters as above we get the whole set of equations $\phi_j = l_j$. The function ϕ_a repeats itself in l_a' which is therefore by this process a function of all the parameters including l_a; presumably the rest of the parameters of the transformed function are also functions of all the parameters including l_a. Any one of these transformed parameters as l_b' that contains l_a contains all the parameters; for transforming $\phi_b = l_b$ by U we get again $l_a' = l_a$. But by this process l_a' is a function only of the parameters in ϕ_b and the elements of M. Since we know that l_a contains all the parameters, ϕ_b must contain all of them. In like manner every function $\phi_a, \phi_b, \ldots \phi_e$ in the set that contains l_a contains all the parameters, and from each of these functions can be generated all the functions of the set ϕ_j. The set of equations $\phi_j = l_j$ is therefore a self-generating set and no equation not already in the set can be generated from any equation of the set. The system of collineations defined by $\phi_j = l_j$ therefore has the first group property.

Next let us suppose that ϕ_a contains only i, $i<j$, of the parameters l_j. From ϕ_a we are able to generate a set of i of the equations $\phi_j = l_j$. No equation in the set $\phi_j = l_j$ containing a parameter not in ϕ_a can be generated from the i equations generated from ϕ_a. This smaller set of i equations is a self-generating set and also defines a system of collineations having the first group property.

Hence the set of equations $\phi_j = l_j$ is a self-generating set or contains within itself one or more self-generating sets $\phi_i = l_i$. The system of collineations defined by $\phi_i = l_i$ has the first group property; and if the set of relations R is identical with this subset $\phi_i = l_i$, the two systems of collineations are the same. But if the set R contains more relations than $\phi_i = l_i$, then the system of collineation defined by the subset is the larger and contains as a subsystem the system of collineations defined by the set R. We shall confine our attention for the present to the larger system of collineations defined by $\phi_i = l_i$. We can now state the following theorem:

THEOREM 8. The set of relations R which defines a system of collineations within G_s, having the first group property, contains a subset in the form of a set of functions of the elements of the matrix M, each equated to a constant, and each of these constants occurs among the parameters of the functions.

185. *The Functions* ϕ_i. Our next concern is to deduce the properties of the functions ϕ_i. These functions must satisfy two conditions, if the set of equations $\phi_i = l_i$ define a system of collineations having the first group property.

1st, they must be unchanged in form, or as we shall say automorphic, under the linear transformation U (or V); and

2d, the parameters of the transformed functions must reproduce in some order the original functions and no others.

The theory of linear transformation leads us at once to the most general class of functions that satisfy the first condition, viz., that are automorphic under linear transformation. They must be functions only of homogeneous polynomials in the elements of M. Since our transformation U (or V) involves three sets of three variables each, we see that our functions ϕ_i may contain homogeneous polynomials in one, two, or three sets of three variables each. For example, these may be homogeneous linear, quadratic, cubic, etc., forms in one set of variables; bilinear, quadrato-linear, etc., in two sets; or trilinear, etc., in three sets of variables. From the theory of linear transformation we are led to the following general statement: A necessary and sufficient condition that a set of functions ϕ_i shall be invariant in form under a linear transformation T in three cogredient sets of three variables each is that each function of the set be a function only of complete* homogeneous polynomials in one or more of these sets of variables.

186. *The Second Condition.* The second condition, that the constants of the transformed functions shall reproduce the original functions, gives us at once the specific form of the functions ϕ_i. They can be none other than complete homogeneous polynomials in the elements of the rows or columns of M; for the constants in the transformed functions are the coefficients of the powers and products of α_1, β_1, etc., in the transformed polynomials entering into the functions, and these coefficients are complete homogeneous polynomials in the elements of the rows or columns of M. An illustration will make this clearer. Suppose, for example, that ϕ is an exponential function of a linear homogeneous polynomial in the elements of one row of M_2; thus $e^{lA_1+mB_1+nC_1}$. We then have the three forms $e^{la_1+mb_1+nc_1}$, $e^{l\alpha_1+m\beta_1+n\gamma_1}$, $e^{lA_1+mB_1+nC_1}$. Making the substitutions U_1 in the last form we get

$$e^{(la_1+mb_1+nc_1)\alpha_1+(la_2+mb_2+nc_2)\beta_1+(la_3+mb_3+nc_3)\gamma_1};$$

---

* By a complete homogeneous polynomial in a given number of variables we mean one that contains all the terms consistent with the number of variables and the degree of the polynomial. Such and only such polynomials are invariant in form under a general linear transformation in all the variables of the set.

but this must be identical with $e^{l\alpha_1+m\beta_1+n\gamma_1}$; equating parameters in these two forms we get

$$\begin{aligned}la_1+mb_1+nc_1&=l,\\ la_2+mb_2+nc_2&=m,\\ la_3+mb_3+nc_3&=n;\end{aligned}$$

we do not get by this process the form $e^{la_1+mb_1+nc_1}$. It is easy to see that the only forms of the functions ϕ_i which reproduce themselves by this process are complete homogeneous polynomials in the elements of M.

187. *The Polynomials* ϕ_i. The homogeneous polynomials ϕ_i are not the most general form of such polynomials, but are restricted to such forms as occur in the coefficients of the transformed forms ϕ_i'. In order to determine this restriction we must examine more closely the phenomena of linear transformation of homogeneous forms in three cogredient sets of three variables each.

Let f_i be a set of i homogeneous polynomials in one, two, or three sets of three variables each; and let each polynomial of the set be transformed by a linear transformation T with matrix M' in three cogredient sets of variables; it is required that the coefficients of the transformed polynomials shall reproduce in the elements of the matrix of T the original set of polynomials and no others.

Our method of procedure is as follows: We choose from the set of functions f_i any one f of the set and transform this by T; the coefficients of the transformed form f' are also functions of the set f_i and so we join these to f and call the new system thus found f_j. We now transform the system f_j by T and join to f_j the new functions appearing among the coefficients of the transformed forms; we continue this process until the system closes. We then have the system of functions f_i.

Let us choose from the set f_i any function f of the set; f will be a homogeneous function of some degree r either in one, two, or three sets of variables; for only polynomials of these three types occur in f_i. Let f be transformed by T into f';

no matter to which of the types f belongs, among the coefficients of f' will be found a homogeneous polynomial of degree r in the elements of the first column of M'. Let us replace a_1, a_2, a_3, by x, y, z, respectively in this function; let us call this function f_1 and use it as the generator of the system f_1.

Let f_1 be transformed by T; the following statements hold:

(a) The number of terms in f_1 and therefore in the transformed form f_1' is $\frac{(r+1)(r+2)}{2}$.

(b) The coefficients of the transformed form f_1' constitute a set of $\frac{(r+1)(r+2)}{2}$ homogeneous polynomials each of degree r in the elements of the columns of M' and each contains $\frac{(r+1)(r+2)}{2}$ terms.

(c) The corresponding terms of this set of polynomials all have the same coefficient, viz.: the coefficient of the corresponding term of f_1.

(d) Three of the coefficients of f_1' are homogeneous polynomials in the elements of a single column each of M'.

(e) When $r > 1$, $\frac{r(r+1)}{2}$ of the coefficients of f_1' are homogeneous and symmetrical polynomials in the elements of pairs of columns of M'.

(g) When $r > 1$, $r-2$ of the coefficients of f_1' are homogeneous and symmetrical polynomials in the elements of the three columns of M'.

In the set of $\frac{(r+1)(r+2)}{2}$ polynomials obtained from the coefficients of f_1' we replace the a's by x, y, z; the b's by x', y', z'; the c's by x'', y'', z''; we now have a set of functions f_j which must be contained within the set f_i.

Let f_2 and f_3 be homogeneous and *symmetrical* polynomials of degree r in two and three sets, respectively, of three variables each, and let each of them be linearly transformed by the same T as above. Precisely the same set of statements hold also for each of these polynomials as for the polynomial f_1.

Therefore when the set of functions f_j is transformed by T, the coefficients of each of the transformed functions reproduce precisely the original functions f_j. The process therefore closes and the set of functions f_j is identical with the set f_i.

188. *A Complete Family of Automorphic Forms.* This result is independent of the degree r of the function f with which we started, and it is also independent of the numerical values of the coefficients of f. Such a set of functions f_i with arbitrary coefficients therefore exists for every positive integer r. The set of functions f_i must have the following properties:

1st. Each function of the set must be of the same degree r in the combined three sets of variables.

2d. Each function of the set must be symmetrical in each of the three sets of variables.

3d. The set of functions must contain all the functions obtainable by all possible combinations of the three sets of variables consistent with the degree r, viz.: $\frac{(r+1)\ (r+2)}{2}$.

4th. The corresponding terms of each function of the set must have the same coefficient.

A set of functions having these properties we shall call a *complete family of automorphic forms.*

THEOREM 9. The most general set of functions f_i of degree r in three sets of three variables each which satisfy these two conditions, viz.: (1) that they are automorphic under a linear transformation T in three sets of cogredient variables, and (2) that the coefficients of the transformed functions reproduce just the original functions, is a complete family of automorphic forms of degree r.

189. *Examples of Complete Families.* We give a few examples of complete families of automorphic forms. Let $r = 1$; in this case we have three linear functions of three terms each in three sets of variables, the coefficients of the corresponding terms being the same in all three functions, thus

$$f_{r=1}: \quad \begin{array}{l} lx + my + nz \ , \\ lx' + my' + nz' \ , \\ lx'' + my'' + nz'' \ . \end{array} \qquad (20)$$

Let these functions be transformed by T; the transformed forms f_r' are

$$f_r' : \begin{array}{l} (la_1+ma_2+na_3)\ x + (lb_1+mb_2+nb_3)\ y + (lc_1+mc_2+nc_3)\ z\ , \\ (la_1+ma_2+na_3)\ x' + (lb_1+mb_2+nb_3)\ y' + (lc_1+mc_2+nc_3)\ z'\ , \\ (la_1+ma_2+na_3)\ x'' + (lb_1+mb_2+nb_3)\ y'' + (lc_1+mc_2+nc_3)\ z''\ . \end{array} \qquad (20')$$

We see here that the coefficients of each of the transformed forms reproduce in the elements of the columns of M the original functions.

Again let $r=2$; the six functions of our complete family are as follows:

$$f_{r=2} : \begin{array}{l} lx^2+my^2+nz^2+2pxy+2qxz+2ryz, \\ lx'^2+my'^2+nz'^2+2px'y'+2qx'z'+2ry'z', \\ lx''^2+my''^2+nz''^2+2px''y''+2qx''z''+2ry''z'', \\ lxx'+myy'+nzz'+p\,(xy'+x'y)+q\,(xz'+x'z)+r\,(yz'+y'z), \\ lxx''+myy''+nzz''+p\,(xy''+x''y)+q\,(xz''+x''z)+r\,(yz''+y''z), \\ lx'x''+my'y''+nz'z''+p\,(x'y''+x''y')+q\,(x'z''+x''z')+r\,(y'z''+y''z'). \end{array} \qquad (21)$$

It is easy to verify in this case also that the coefficients of each of the transformed forms reproduce in the elements of the columns of M' the original forms.

In like manner we can write down a complete family of automorphic forms of any degree r and verify the fundamental properties of the family.

190. *The Effect of the Transformations U and V.* We must now return to the consideration of the equations $\phi_i = l_i$ among the elements of M, which define a system of collineations within G, having the first group property; and we shall apply the results of Theorem 9 to these equations.

Let the three sets of cogredient variables be respectively the elements of the three rows of the matrix M_2; let $\phi_i(A, B,$ etc.,) be a complete family of automorphic forms in the elements of the rows of M_2 and let ϕ_i be transformed by U. Since the matrix of U is the conjugate of the matrix of M, the coefficients of each of the transformed forms ϕ_i will reproduce in the elements of the rows of M the original forms in the rows of M_2. Equating corresponding coefficients of the original and transformed forms we have a set of equations $\phi_i(a_1, b_1, \text{etc.},) = l_i$ which satisfies both conditions of Art. 185.

We may let our three sets of cogredient variables be respectively the elements of the three columns of M_1; if the complete family φ_i of automorphic forms in the elements of these columns be transformed by V, the coefficients of each of the transformed forms φ_i will reproduce in the elements of the columns of M_1 the original forms. Equating as before corresponding coefficients of the original and the transformed forms we have a set of equations $\phi_i(\alpha_1, \alpha_2, \text{etc.},) = l_i$ which satisfies both conditions of Art. 185.

We are now able to restate Theorem 8 in a more precise form, thus:

THEOREM 8*a*. The set of equations $\varphi_i = l_i$ on the elements of the matrix M, which defines a system of collineations within G_8 having the first group property, consists of a complete family of automorphic forms in the elements of the rows or columns of M, each member of the family equated to the corresponding coefficient of the family.

191. *The Linear Families.* If our complete family of automorphic forms is linear in the rows of M, the equations $\varphi_i = l_i$ are

$$R_p: \quad \begin{aligned} la_1 + mb_1 + nc_1 &= l, \\ la_2 + mb_2 + nc_2 &= m, \\ la_3 + mb_3 + nc_3 &= n, \end{aligned} \tag{22}$$

where l, m, and n, are arbitrary constants and not all zero. It is an easy matter to write these equations in terms of A_1, etc., transform them by U, equate coefficients of corresponding terms and get the same equations in a_1, b_1, etc.

A convenient method of verifying the sufficiency of these conditions is as follows: Write down these relations on the elements of M and M_1, thus:

$$\begin{aligned} la_1 + mb_1 + nc_1 &= l, \\ la_2 + mb_2 + nc_2 &= m, \\ la_3 + mb_3 + nc_3 &= n, \end{aligned} \quad \text{and} \quad \begin{aligned} l\alpha_1 + m\beta_1 + n\gamma_1 &= l, \\ l\alpha_2 + m\beta_2 + n\gamma_2 &= m, \\ l\alpha_3 + m\beta_3 + n\gamma_3 &= n \end{aligned}$$

Substitute for l, m, and n, on the left-hand side of the second set their values from the first set and collect, we thus get

$$\begin{aligned} lA_1 + mB_1 + nC_1 &= l, \\ lA_2 + mB_2 + nC_2 &= m, \\ lA_3 + mB_3 + nC_3 &= n. \end{aligned} \tag{22'}$$

Hence if these relations are imposed upon the elements of M and M_1, they are also satisfied by the elements of M_2.

In addition to the relations of equations (22) there may be other relations at the same time on the elements of M, M_1, and M_2. But the relations (22) are alone sufficient to insure that the system of collineations satisfying them has the first group property.

A second family linear in the columns of M is

$$R_l \ : \quad \begin{array}{l} l a_1 + m a_2 + n a_3 = l\,, \\ l b_1 + m b_2 + n b_3 = m\,, \\ l c_1 + m c_2 + n c_3 = n. \end{array} \qquad (23)$$

The sufficiency of these relations may be shown in a manner similar to that for R_p.

192. *The Quadratic Families R_2.* If $r = 2$ in our complete family of automorphic forms, we can write down two sets of quadratic relations $\phi_i = l_i$ on the elements of M, one in the rows and the other in the columns of M. The set in the columns of M is as follows:

$$R_2 \ : \quad \begin{array}{l} l a_1^2 + m a_2^2 + n a_3^2 + 2p\, a_1 a_2 + 2q\, a_1 a_3 + 2r\, a_2 a_3 = l\,, \\ l b_1^2 + m b_2^2 + n b_3^2 + 2p\, b_1 b_2 + 2q\, b_1 b_3 + 2r\, b_2 b_3 = m\,, \\ l c_1^2 + m c_2^2 + n c_3^2 + 2p\, c_1 c_2 + 2q\, c_1 c_3 + 2r\, c_2 c_3 = n\,, \\ l a_1 b_1 + m a_2 b_2 + n a_3 b_3 + p\,(a_1 b_2 + a_2 b_1) + q\,(a_1 b_3 + a_3 b_1) + r\,(a_2 b_3 + a_3 b_2) = p, \\ l a_1 c_1 + m a_2 c_2 + n a_3 c_3 + p\,(a_1 c_2 + a_2 c_1) + q\,(a_1 c_3 + a_3 c_1) + r\,(a_2 c_3 + a_3 c_2) = q, \\ l b_1 c_1 + m b_2 c_2 + n b_3 c_3 + p\,(b_1 c_2 + b_2 c_1) + q\,(b_1 c_3 + b_3 c_1) + r\,(b_2 c_3 + b_3 c_2) = r. \end{array} \qquad (24)$$

The system of collineations defined by these relations has the first group property. The sufficiency of the conditions may be shown by actually transforming the relations in A_1, B_1, etc., by V and equating coefficients; or by substituting as above the relations on the elements of M_1 in those of M.

These examples suffice to illustrate the general theorem; the same consideration may be extended to complete families of any degree.

193. *Other Expressions for ϕ_i.* We have thus far only one method of expressing a complete family of automorphic forms which may be stated precisely as follows: When a complete homogeneous polynomial of degree r in one set of three variables is transformed by a linear transformation T,

the coefficients of the transformed form are polynomials in the elements of the matrix of T; these constitute a complete family of automorphic forms of degree r.

A complete family of automorphic forms may be expressed in another very convenient manner. Let us take two linear transformations T and T' such that the matrix of T' is the conjugate of that of T. Thus

$$T\ :\quad \begin{array}{l} x=a_1x+b_1y+c_1z\,, \\ y=a_2x+b_2y+c_2z\,, \\ z=a_3x+b_3y+c_3z\,, \end{array} \quad \text{and } T'\ :\quad \begin{array}{l} x=a_1x+a_2y+a_3z\,, \\ y=b_1x+b_2y+b_3z\,, \\ z=c_1x+c_2y+c_3z\,. \end{array} \qquad (25)$$

A complete family of linear automorphic forms in the rows (or columns) of M may be obtained by replacing the variables $x, y, z,$ of T (or T') by $l, m, n,$ respectively.

A complete family of quadratic automorphic forms in the rows (or columns) of M is obtained by squaring the three equations of T (or T') and forming their products two at a time and then replacing x^2 by l, y^2 by m, z^2 by n, xy by p, etc. A complete family of cubic automorphic forms is obtained in an analogous manner and the process holds for the family of degree r.

194. *Determinant of an Automorphic Family.* Associated with every complete family of automorphic forms is a certain determinant which requires attention. The determinant of a linear automorphic family is equal to Δ the determinant of T or T' (25). It is evident from the last mode of expression for a complete family that the determinant of the family cannot be independent of Δ. It is not difficult to prove the following identity:

$$\left\| \begin{array}{cccccc} a_1^2 & b_1^2 & c_1^2 & 2a_1b_1 & 2a_1c_1 & 2b_1c_1 \\ a_2^2 & - & - & - & - & - \\ a_3^2 & - & - & - & - & - \\ a_1a_2 & - & - & - & - & - \\ a_1a_3 & - & - & - & - & - \\ a_2a_3 & - & - & - & - & - \end{array} \right\| \equiv \Delta^4\,.$$

In like manner the determinant of a cubic family is equal to Δ^9; and in general the determinant of an automorphic family of degree r is equal to Δ to the power r^2.

195. *The Second Group Property.* It remains to be shown that the system of collineation satisfying a set of relations $\phi_i = l_i$ also has the second group property, viz.: that the inverse of every collineation in the set is also in the set.

We shall begin by establishing the second group property for the set of linear relations given in equations (22). A collineation T and its inverse T^{-1} are as follows:

$$T: \quad \begin{array}{l} x_1 = a_1 x + b_1 y + c_1 z, \\ y_1 = a_2 x + b_2 y + c_2 z, \\ z_1 = a_3 x + b_3 y + c_3 z, \end{array} \qquad T^{-1}: \quad \begin{array}{l} \triangle x = A_1 x_1 + A_2 y_1 + A_3 z_1, \\ \triangle y = B_1 x_1 + B_2 y_1 + B_3 z_1, \\ \triangle z = C_1 x_1 + C_2 y_1 + C_3 z_1. \end{array}$$

The coefficients of T satisfy the relations

$$R_1: \quad \begin{array}{l} l a_1 + m b_1 + n c_1 = l, \\ l a_2 + m b_2 + n c_2 = m, \\ l a_3 + m b_3 + n c_3 = n. \end{array} \qquad (22)$$

If the coefficients of T^{-1} also satisfy R_1, we must have

$$R'_1: \quad \begin{array}{l} l A_1 + m A_2 + n A_3 = \triangle l, \\ l B_1 + m B_2 + n B_3 = \triangle m, \\ l C_1 + m C_2 + n C_3 = \triangle n. \end{array} \qquad (26)$$

But equations R_1' may be derived directly from R_1 as follows: Multiplying the first equation of R_1 by A_1, the second by A_2, the third by A_3 and adding, we get the first equation of R_1'. In like manner by multiplying by the B's and C's we get the other equations of R_1'. Hence the system of collineations satisfying the relations R_1 has both group properties and is therefore a group.

The same process may be applied to the set of quadratic relations given in (24). If we multiply the six equations of (24) respectively by A_1^2, B_1^2, C_1^2, $2A_1B_1$, $2A_1C_1$, $2B_1C_1$, and add, we get

$$lA_1^2 + mB_1^2 + nC_1^2 + 2pA_1B_1 + 2qA_1C_1 + 2rB_1C_1 = \Delta^2 l.$$

In like manner we obtain the whole set of relations

$$R_2': \quad \begin{array}{l} lA_1^2 + mB_1^2 + \ - \ - \ - \ - \ = \triangle^2 l, \\ lA_2^2 + \ - \ - \ - \ - \ - \ = \triangle^2 m. \\ lA_3^2 + \ - \ - \ - \ - \ - \ = \triangle^2 n, \\ lA_1A_2 + \ - \ - \ - \ - \ - \ = \triangle^2 p, \\ lA_1A_3 + \ - \ - \ - \ - \ - \ = \triangle^2 q, \\ lA_2A_3 + \ - \ - \ - \ - \ - \ = \triangle^2 r. \end{array} \qquad (27)$$

But these relations show that T^{-1} satisfies equations (24).

Hence the system of collineations satisfying equations (24) has both group properties.

This method is applicable in general to any set of equations $\phi_i = l_i$ where ϕ_i is a complete family of automorphic forms of degree r in the elements of the rows or columns of M. For every set of relations R_r given by $\phi_i = l_i$ there is a corresponding set R_r' given by $\psi_i = \Delta^r l_i$, where ψ_i is obtained from ϕ_i by substituting for the elements of M their cofactors in the matrix conjugate to M. Hence the system of collineations satisfying a set of relations $\phi_i = l_i$ has both group properties.

THEOREM 10. Every set of relations $\phi_i = l_i$, where ϕ_i is a complete family of automorphic forms in the elements of the rows (or columns) of M, implies another set $\psi_i = \Delta^r l_i$ where ψ_i is a complete family of automorphic forms in the elements of the columns (or rows) of M^{-1}. The first set of relations $\phi_i = l_i$ establishes for the system of collineations satisfying them the first group property; the second set $\psi_i = \Delta^r l_i$ establishes for the system the second group property.

196. *Analytic Conditions for a Subgroup of G_8.* We are now in position to determine the form which the relations R must assume in order to define a subgroup of G_8. We have shown in Theorem 9 that the functions ϕ_i must constitute a complete family of automorphic forms of degree r. It follows that among the relations R which define a subgroup of G_8 there must be a complete family of automorphic forms each equated to the corresponding coefficient of its family.

THEOREM 11. A necessary and sufficient condition for the existence of a subgroup of G_8 is that the elements of the matrix M satisfy a set of equations $\phi_i = l_i$ consisting of a complete family of automorphic forms in the elements of the rows or columns of M, each equated to the corresponding coefficient of the family.

§ 4. Groups of Type I Determined by Linear and Quadratic Relations.

A. SUBGROUPS OF G_8; GEOMETRIC METHOD.

We shall now attack the problem of the determination of all varieties of subgroups of G_8 of type I which are defined by a linear or quadratic set of relations R. Before taking up the general analytic method of solving this problem it is desirable to consider it synthetically and give, as it were, a geometrical forecast of the principal results. The geometrical method is lacking in rigor but is valuable for the light which it throws on the problem.

197. *Number of Collineations of Type I.* We have seen that every collineation of type I leaves a triangle invariant and is further characterized by two independent cross-ratios k and k'. Every collineation of type I depends therefore upon eight constants, viz., the six coordinates of the three vertices of the invariant triangle and these two independent cross-ratios. Since each of these constants may assume ∞^1 different values, we see that there are ∞^8 collineations of type I in the plane. We are also enabled to distinguish two distinct kinds of variable parameters, viz., coordinates of invariant points and characteristic cross-ratios.

If we suppose the vertices of the invariant triangle to remain the same but let the cross-ratios k and k' vary through all possible values, we get a system of ∞^2 different collineations all having the same invariant triangle. By Theorem 1 of this chapter these form a group which may be designated by $G_2(AA'A'')$, the two parameters being k and k'.

Since there are ∞^6 triangles in the plane, it follows that there are ∞^6 such two-parameter groups as $G_2(AA'A'')$ in the plane. Hence, the group of all collineations in the plane G_8 contains ∞^6 two-parameter subgroups $G_2(AA'A'')$. No two of these two-parameter groups can have a collineation of type I in common; for, if two collineations T and T_1 are the same, the eight parameters of the one must be equal to the eight parameters of the other; but if the collineations leave different triangles invariant, all of the six coordinate parameters of the one cannot be equal to those of the other. Hence T and T_1 cannot be the same except when both are identical collineations.

198. *Subgroups of G_8.* By the aid of the principle that all collineations which leave a certain figure invariant form a group, it is not difficult to enumerate all varieties of subgroups of G_8 that can be compounded out of the ∞^6 two-parameter groups $G_2(AA'A'')$ in the plane. It is only necessary to recall all configurations of lines and points that can be made up of triangles. They are as follows: a line l, a point A, a pair of lines ll' (and their intersection), a pair of points AA' (and their join), a lineal element Al, a point A and a line l not through A, two points AA' their join and a line l through one of the points, three points or three lines forming a triangle. These eight configurations (shown in Fig. 24) are the invariant figures of subgroups of G_8, as follows: $G_6(l)$, $G_6(A)$, $G_5(Al)$, $G_4(AA')$, $G_4(ll')$, $G_4(A,l'')$, $G_3(AA'l')$, $G_2(AA'A'')$. We shall briefly discuss each variety of group in detail.

199. *The Groups $G_6(l)$ and $G_6(A)$.* There are ∞^8 collineations of type I in the plane and only ∞^2 lines in the plane; hence, any line of the plane can be transformed into any other line or into itself in ∞^6 different ways. The ∞^6 collineations, which transform a line l into itself, form a six-parameter group $G_6(l)$. There are ∞^4 triangles having the side l in common; each of these triangles is the invariant triangle of a two-parameter group $G_2(AA'A'')$. Thus we see that $G_6(l)$ is made up of ∞^4 two-parameter groups $G_2(AA'A'')$.

In like manner, we see that any point A of the plane is the invariant figure of a six-parameter group $G_6(A)$. Each of the ∞^4 triangles having A for one vertex is the invariant tri-

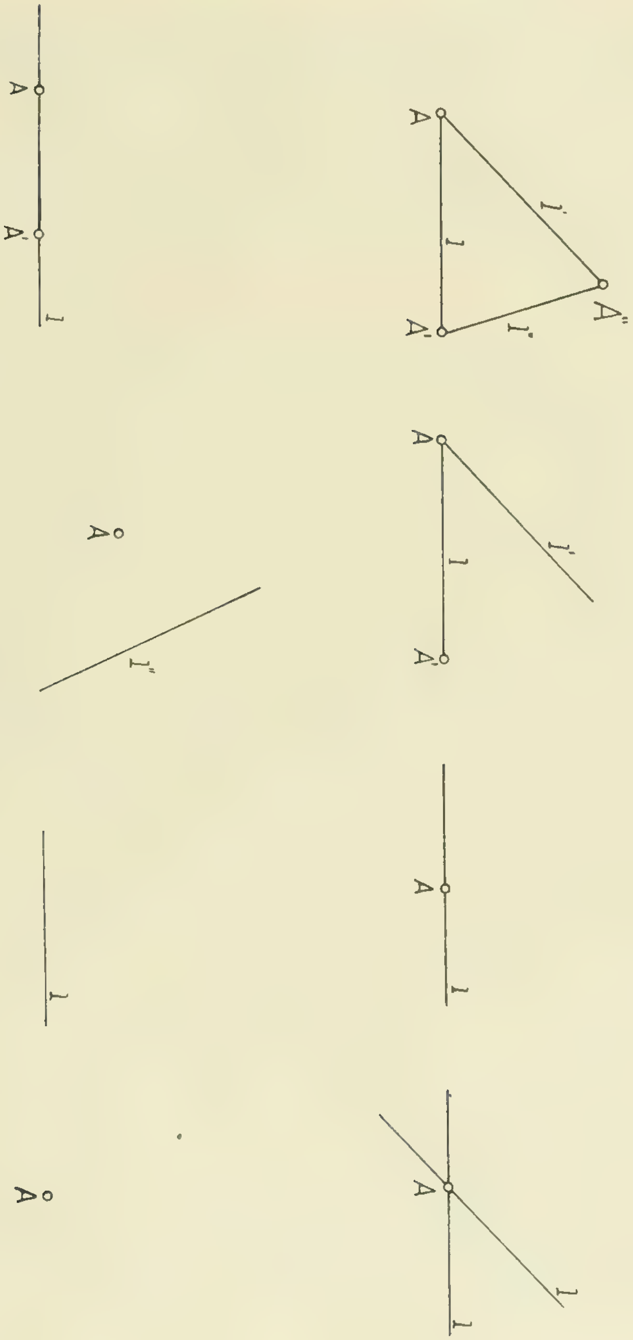
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FIG. 24.

angle of a two-parameter subgroup of $G_6(A)$. The groups $G_6(l)$ and $G_6(A)$ are dualistic groups. G_8 contains ∞^2 subgroups $G_6(l)$ and also ∞^2 subgroups $G_6(A)$.

Each of these groups contains, besides collineations of type I, also collineations of other types; but the collineations composing a group $G_6(l)$ or $G_6(A)$ are chiefly of type I. The structure of all these groups will be examined in detail in Chapter IV.

200. *The Group* $G_5(Al)$. The general projective group G_8 contains ∞^3 subgroups $G_5(Al)$, one for each lineal element in the plane. Each group $G_5(Al)$ is made up of ∞^3 two-parameter groups $G_2(AA'A'')$ whose invariant triangles have one vertex and one side in common. Each group $G_6(l)$ contains ∞^1 subgroups $G_5(Al)$, one for each point on l; likewise each group $G_6(A)$ contains ∞^1 subgroups $G_5(Al)$, one for each line through A. The group $G_5(Al)$ is self-dualistic.

201. *The Groups* $G_4(AA')$ *and* $G_4(ll')$. Two points AA' and their join l form the invariant figure of a four-parameter group $G_4(AA')$. The group G_8 contains ∞^4 equivalent subgroups $G_4(AA')$, one for each pair of points in the plane. A group $G_4(AA')$ is composed of ∞^2 two-parameter groups $G_2(AA'A'')$ whose invariant triangles have the vertices A and A' in common.

In like manner, G_8 contains ∞^4 equivalent subgroups $G_4(ll')$, one for each pair of lines in the plane. A group $G_4(ll')$ is composed of ∞^2 two-parameter groups $G_2(AA'A'')$ whose invariant triangles have one vertex and two sides in common. The groups $G_4(AA')$ and $G_4(ll')$ are dualistic. $G_6(l)$ contains ∞^2 subgroups $G_4(AA')$, also ∞^2 subgroups $G_4(ll')$; $G_6(A)$ breaks up in a similar manner.

202. *The Group* $G_4(A, l'')$. There are ∞^4 combinations of point and line in a plane where the point is not on the line. Each combination of point and line is the invariant figure of a four-parameter group $G_4(A, l'')$. The group $G_4(A, l'')$ is composed of ∞^2 two-parameter groups $G_2(AA'A'')$, whose in-

variant triangles have a vertex and the opposite side in common. Thus we see that G_8 contains ∞^4 equivalent four-parameter sub-groups $G_4(A, l'')$; these groups are self-dualistic.

203. *The Group* $G_3(AA'l')$. Triangles ∞^1 in number can be arranged with two of their vertices at A and A' and their third vertex on a line Al'. Each of these triangles is the invariant triangle of a two-parameter group $G_2(AA'A'')$; these ∞^1 two-parameter groups unite to form a three-parameter group $G_3(AA'l')$.

The group of all collineations in the plane G_8 contains ∞^5 equivalent subgroups $G_3(AA'l')$; a group of this kind is self-dualistic. The group $G_4(AA')$ contains ∞^1 subgroups of the kind $G_3(AA'l')$, one for each line through A; also one such group for each line through A'. The group $G_4(ll')$ contains ∞^1 subgroups of the kind $G_3(AA'l')$, one for each point on l', and also one for each point on l.

204. *The Group* $G_3(K)$. A collineation transforms a conic into a conic. Since there are ∞^5 conics and ∞^8 collineations in the plane, we infer that each conic of the plane may be transformed into itself in ∞^3 ways. These ∞^3 collineations leaving the conic K invariant form a group $G_3(K)$. Any collineation of the group transforms the points on K into points on K and tangents to K into tangents to K. Also a lineal element of the conic K, consisting of a point on K and the tangent to K at the point, is transformed by T into a lineal element of K. In particular, if a point on K is invariant the tangent to K at A is also invariant.

A conic K and a point A, not on K, may be simultaneously invariant under ∞^1 collineations; for there are only ∞^7 such combinations in the plane. We therefore infer the existence of a one-parameter group $G_1(A, K)$. In like manner we should expect to find a one-parameter group $G_1(l, K)$, leaving invariant a line l and a conic K. When A and K are both invariant and A is not on K, the two tangents from A to K are both invariant lines; hence their points of contact are also invariant points and their join is another invariant line. The invariant figure consists therefore of a triangle $(AA'A'')$ and a conic K having two of the sides as tangents and the third side as chord of contact. When K and l are both invariant and l does not touch K, then the two points of intersection of l and K are also invariant; hence the tangents at these two points are also invariant and their intersection is another invariant point. Thus we have the same figure as before and there is but one such variety of group, $G_1(AA'A''K)$.

When A is on K, the tangent l at A is also invariant; this case determines a two-parameter group $G_2(AlK)$.

THEOREM 12. The General Projective Group G_8 has at least eleven varieties of subgroups of type I; these are $G_6(A)$, $G_6(l)$, $G_5(Al)$, $G_4(AA')$, $G_4(ll')$, $G_4(A,l'')$, $G_3(AA'l')$, $G_2(AA'A'')$; $G_3(K)$, $G_2(AlK)$, $G_1(AA'A''K)$.

B. GROUPS DEFINED BY LINEAR AND QUADRATIC RELATIONS.

We shall now return to the analytic point of view and apply the results reached in Theorems 9 and 11 to the solution of the problem of finding all varieties of subgroups of G_8 of type I defined by linear and quadratic relations on the elements of the matrix M.

205. *Linear Functions of the Elements of the Rows.* Let the complete family of automorphic forms be linear functions in the elements of the rows of M. Our set of relations R are of the form

$$R_p\ :\quad \begin{array}{l} la_1+mb_1+nc_1=l\ , \\ la_2+mb_2+nc_2=m, \\ la_3+mb_3+nc_3=n\ , \end{array} \qquad (22)$$

and these define the group given in the illustrative example of Art. 191. The form of these relations shows us at once that the ratios of three numbers l, m, n, are absolutely invariant under all the transformations of the group.

The geometric invariant of the group is also evident from the form of equations (22). These show that the point whose coordinates are proportioned to l, m, n, is invariant under all transformations of the group. This is a six-parameter group, since the three relations among the nine elements of M leave six independent parameters. It will be designated by $G_6(A)$.

206. *Linear Functions of the Elements of the Columns.* Let ϕ_i be linear homogeneous functions of the elements of the columns of M. Then by Art. 191 we have three relations of the form

$$R_l: \quad \begin{aligned} la_1+ma_2+na_3&=l, \\ lb_1+mb_2+nb_3&=m, \\ lc_1+mc_2+nc_3&=n, \end{aligned} \tag{23}$$

i. e., the ϕ function of the elements of each column is equated to the coefficient of the elements of the corresponding row. Relations of this form satisfy the necessary conditions for a subgroup of G_8.

In order to show that this set of relations is sufficient to define a subgroup of G_8 we assume that they hold for T and T_1; thus

$$\begin{aligned} la_1+ma_2+na_3&=l, & l\alpha_1+m\alpha_2+n\alpha_3&=l, \\ lb_1+mb_2+nb_3&=m, & l\beta_1+m\beta_2+n\beta_3&=m, \\ lc_1+mc_2+nc_3&=n, & l\gamma_1+m\gamma_2+n\gamma_3&=n. \end{aligned}$$

Substituting for l, m, n, on the left hand side of the first set their values from the second set, we get

$$\begin{aligned} lA_1+mA_2+nA_3&=l, \\ lB_1+mB_2+nB_3&=m, \\ lC_1+mC_2+nC_3&=n. \end{aligned} \tag{23'}$$

These relations hold therefore for T_2 and they are sufficient to define a group.

If equations (23) are multiplied respectively by x, y, and z, and then added we get

$$\begin{aligned} f\equiv lx+my+nz=l\,(a_1x+b_1y+c_1z)\ +m\,(a_2x+b_2y+c_2z)\ +n\,(a_3x+b_3y+c_3z), \\ =lx_1+my_1+nz_1. \end{aligned}$$

This shows that the function $f\equiv lx+my+nz$ is invariant in form under all the transformations of the group and that this function has the same value at a pair of corresponding points of the plane. This function vanishes at all points of a certain line of the plane and hence the geometric invariant of the group is the line l whose equation is

$$lx+my+nz=0.$$

The group is a six-parameter group and will be designated by $G_6(l)$.

207. *Most General Linear Function of the Elements.* Let us now assume the most general possible linear relation among the elements of M_2, thus

$$\phi \equiv lA_1 + mA_2 + nA_3 + pB_1 + qB_2 + rB_3 + sC_1 + tC_2 + uC_3 = v.$$

Substitute for A_1, B_1, etc., their values from M_2 and set the coefficients of α_1, β_1, etc., equal to the coefficients A_1, B_1, etc.; we thus obtain the following nine relations:

$$\begin{array}{lll} la_1 + pb_1 + sc_1 = l, & ma_1 + qb_1 + tc_1 = m, & na_1 + rb_1 + uc_1 = n, \\ la_2 + pb_2 + sc_2 = p, & ma_2 + qb_2 + tc_2 = q, & na_2 + rb_2 + uc_2 = r, \\ la_3 + pb_3 + sc_3 = s, & ma_3 + qb_3 + tc_3 = t, & na_3 + rb_3 + uc_3 = u. \end{array} \qquad (28)$$

Our one assumed relation among the elements of M_2 leads to nine relations among the elements of M and evidently there is no group corresponding to our assumed relation.

But $\phi = v$ may be considered as the sum of three independent relations as follows:

$$lA_1 + pB_1 + sC_1 = l, \quad mA_2 + qB_2 + tC_2 = q, \quad nA_3 + rB_3 + uC_3 = u.$$

Each of these relations is in the form of a linear homogeneous function of the elements of a row of M_2 equated to a constant; and we may therefore write down at once the three sets of relations, thus:

$$\begin{array}{lll} lA_1 + pB_1 + sC_1 = l, & mA_1 + qB_1 + tC_1 = m, & nA_1 + rB_1 + uC_1 = n, \\ lA_2 + pB_2 + sC_2 = p, & mA_2 + qB_2 + tC_2 = q, & nA_2 + rB_2 + uC_2 = r, \\ lA_3 + pB_3 + sC_3 = s, & mA_3 + qB_3 + tC_3 = t, & nA_3 + rB_3 + uC_3 = u. \end{array} \qquad (28')$$

Applying to these three sets of relations the same process we applied to $\phi = v$ we get the same nine relations as before and the correspondence is complete. We thus have the conditions for a group.

These nine relations taken three at a time are equivalent to only six independent relations among eight independent parameters; hence they define a two-parameter group G_2. The geometric invariant of this group is readily seen by considering the three relations down the first column of (28). These show that the point (l, p, s) is transformed into itself. In like manner we see that the points (m, q, t) and (n, r, u) are also transformed into themselves. If the determinant

$$\Delta(l) \equiv \begin{vmatrix} l & m & n \\ p & q & r \\ s & t & u \end{vmatrix} \neq 0,$$

the group leaves invariant a triangle and may properly be designated by $G_2(AA'A'')$.

When the relation $\phi = v$ is transformed by substituting for A_1, B_1, etc., their values, we may collect in a different manner and equate the coefficients of a_1, b_1, etc., to the coefficients of A_1, B_1, etc., in ϕ; this gives us a set of nine relations in the elements of M_1 as follows:

$$\begin{array}{lll} l\alpha_1+m\alpha_2+n\alpha_3=l, & p\alpha_1+q\alpha_2+r\alpha_3=p, & s\alpha_1+t\alpha_2+u\alpha_3=s, \\ l\beta_1+m\beta_2+n\beta_3=m, & p\beta_1+q\beta_2+r\beta_3=q, & s\beta_1+t\beta_2+u\beta_3=t, \\ l\gamma_1+m\gamma_2+n\gamma_3=n, & p\gamma_1+q\gamma_2+r\gamma_3=r, & s\gamma_1+t\gamma_2+u\gamma_3=u. \end{array} \tag{29}$$

If we assume the nine relations

$$\begin{array}{lll} lA_1+mA_2+nA_3=l, & pA_1+qA_2+rA_3=p, & sA_1+tA_2+uA_3=s, \\ lB_1+mB_2+nB_3=m, & pB_1+qB_2+rB_3=q, & sB_1+tB_2+uB_3=t, \\ lC_1+mC_2+nC_3=n, & pC_1+qC_2+rC_3=r, & sC_1+tC_2+uC_3=u, \end{array} \tag{29'}$$

and transform them as before we get the same nine relations in the elements of the columns of M. Thus we have again the conditions for a two-parameter group. The geometric invariant of this group is the set of three lines whose equations are $lx+my+nz=0$, $px+qy+rz=0$, $sx+ty+uz=0$. If the determinant

$$\Delta(l) \equiv \begin{vmatrix} l & m & n \\ p & q & r \\ s & t & u \end{vmatrix} \neq 0,$$

the group leaves a triangle invariant and may be designated by $G_2(ll'l'')$.

208. *Implied Linear Relations.* If we have given two sets of linear relations of the same kind, say R_p and $R_{p'}$, we can derive from these another set of linear relations of the other kind R_l. Let R_p and $R_{p'}$ be as follows:

$$R_p: \begin{array}{l} la_1+mb_1+nc_1=l, \\ la_2+mb_2+nc_2=m, \\ la_3+mb_3+nc_3=n, \end{array} \quad R_{p'}: \begin{array}{l} l'a_1+m'b_1+n'c_1=l', \\ l'a_2+m'b_2+n'c_2=m', \\ l'a_3+m'b_3+n'c_3=n'. \end{array} \tag{22}$$

Multiply the second equation of R_p by the third of $R_{p'}$, the second of $R_{p'}$ by the third of R_p, and subtract; we thus get

$$L''A_1+M''B_1+N''C_1=L'',$$

where A_1, B_1, etc., are cofactors of a_1, b_1, etc., in M and L'', etc., are cofactors of l'', etc., in

$$\begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix}.$$

In like manner we get the entire set

$$R' : \quad \begin{aligned} &L''A_1+M''B_1+N''C_1=L'', \\ &L''A_2+M''B_2+N''C_2=M'', \\ &L''A_3+M''B_3+N''C_3=N''. \end{aligned} \tag{30}$$

From these three equations we readily derive the set

$$R_L : \quad \begin{aligned} &L''a_1+M''a_2+N''a_3=\triangle L'', \\ &L''b_1+M''b_2+N''b_3=\triangle M'', \\ &L''c_1+M''c_2+N''c_3=\triangle N''. \end{aligned} \tag{31}$$

This secures the invariance of the line $L''x+M''y+N''z=0$, which is the line joining the two points (l, m, n) and (l', m', n').

In the same way two sets R_l and $R_{l'}$ imply the existence of a set of the other kind R_p, which means that the point of intersection of the invariant lines l and l' is an invariant point P.

The three relations R_p, $R_{p'}$, $R_{p''}$, which define the group $G_2(AA'A'')$, taken two at a time give us three other relations R_l, $R_{l'}$, $R_{l''}$, which define the group $G_2(ll'l'')$. Hence these two groups $G_2(AA'A'')$ and $G_2(ll'l'')$ are equivalent groups.

Two relations of different kinds R_p and R_l may be wholly independent of each other or a relation may exist between them. The invariant point (l, m, n) will lie on the invariant line $\lambda x+\mu y+\nu z=0$, if

$$S\lambda l\equiv\lambda l+\mu m+\nu n=0.$$

209. *Subgroups Defined by Linear Relations.* We are now in position to enumerate all the subgroups of G_8 of type I that are defined by linear relations among the elements of the matrix M. It is clear that we may have groups defined by one, two, or three sets of relations of the kind R_p or R_l and any combination of R_p's and R_l's that does not impose too

many conditions on the elements of M. If a relation $S\lambda l=0$ exists between the constants of R_p and R_l, the number of independent conditions on M is decreased by one. The number of varieties of subgroups of G_8 defined by linear relations is therefore a definite finite number, since the number of such combinations is a definite finite number. The following groups are defined each by its proper set of relations:

$G_6(A)$ defined by R_p.
$G_6(l)$ " " R_l.
$G_5(Al)$ " " R_p, R_l and $S\lambda l=0$.
$G_4(A, l'')$ " " R_p and R_l.
$G_4(AA')$ " " R_p and $R_{p'}$.
$G_4(ll')$ " " R_l and $R_{l'}$.
$G_3(AA'l')$ " " R_p, $R_{p'}$, R_l and $S\lambda l=0$ or $S\lambda l'=0$.
$G_2(AA'A'')$ " " R_p, $R_{p'}$ and $R_{p''}$.

This list is in exact agreement with the list given in Theorem 12 except as to the groups $G_3(K)$, $G_2(AlK)$ and $G_1(AA'A''K)$ which remain to be investigated.

We are able to infer from these results that there is no seven-parameter group of plane collineations. The smallest possible number of relations in a set R defining a group is three and the group so defined is a six-parameter group.

THEOREM 13. There is no seven-parameter group of plane collineations.

210. *The Group $G_3(K)$.* We come now to the problem of finding all continuous groups of collineations defined by quadratic relations on the elements of M. There are two possible complete families of quadratic automorphic forms in the elements of M; one of them is homogeneous in the elements of the columns of M and the other in the rows. We shall first consider the former system. The set of relations is

$$R_2:\quad \begin{aligned}
&la_1^2+ma_2^2+na_3^2+2pa_1a_2+2qa_1a_3+2ra_2a_3=l,\\
&lb_1^2+mb_2^2+nb_3^2+2pb_1b_2+2qb_1b_3+2rb_2b_3=m,\\
&lc_1^2+mc_2^2+nc_3^2+2pc_1c_2+2qc_1c_3+2rc_2c_3=n,\\
&la_1b_1+ma_2b_2+na_3b_3+p(a_1b_2+a_2b_1)+q(a_1b_3+a_3b_1)+r(a_2b_3+a_3b_2)=p,\\
&la_1c_1+ma_2c_2+na_3c_3+p(a_1c_2+a_2c_1)+q(a_1c_3+a_3c_1)+r(a_2c_3+a_3c_2)=q,\\
&lb_1c_1+mb_2c_2+nb_3c_3+p(b_1c_2+b_2c_1)+q(b_1c_3+b_3c_1)+r(b_2c_3+b_3c_2)=r.
\end{aligned} \tag{24}$$

This set of relations on the elements of M implies another relation, viz.: that the determinant of M is unity. This may be readily shown as follows: Form the symmetric determinant,

$$D' \equiv \begin{vmatrix} l & p & q \\ p & m & r \\ q & r & n \end{vmatrix},$$

and substitute for these elements their values on the left hand side of R_2; the new determinant factors readily into $\Delta^2 D'$. Whence we have $\Delta^2 = 1$, where Δ is the determinant of M; but by Art. 181 we can take only the positive sign. Hence $\Delta = 1$.

If the six equations of R_2 are multiplied respectively by x^2, y^2, z^2, $2xy$, $2xz$, $2yz$, and added we get

$$f \equiv l(a_1x+b_1y+c_1z)^2+m(a_2x+b_2y+c_2z)^2+ \text{etc.} = lx^2+my^2+nz^2+2pxy+2qxz+2ryz,$$

which shows the function f is invariant under T, and has the same value at a pair of corresponding points of the plane. This function will vanish along the points of some conic K in the plane, which conic is invariant under all the collineations belonging to the group defined by R_2. In order that this be a non-degenerate conic, we must assume $D' \neq 0$.

The six relations R_2 on the nine elements of M leave three independent parameters for the group defined by R_2. This is therefore a three-parameter group and will be designated by $G_3(K)$.

THEOREM 14. The group defined by the quadratic relations R_2 is a three-parameter group and leaves invariant the conic whose equation is

$$lx^2+my^2+nz^2+2pxy+2qxz+2ryz=0.$$

211. *Other Quadratic Relations.* We saw in Art. 195 that the six equations of (24) imply another six, viz.: R'_2 of (27). But since $\Delta = 1$, the equations of R'_2 may be written thus:

$$R'_2: \quad \begin{array}{llllll} lA_1^2+mB_1^2+nC_1^2+2pA_1B_1+2qA_1C_1+2rB_1C_1 = l, \\ lA_2^2+ \quad - \quad - \quad - \quad - \quad - \quad = m, \\ lA_3^2+ \quad - \quad - \quad - \quad - \quad - \quad = n, \\ lA_1A_2+ \quad - \quad - \quad - \quad - \quad - \quad = p, \\ lA_1A_3+ \quad - \quad - \quad - \quad - \quad - \quad = q, \\ lA_2A_3+ \quad - \quad - \quad - \quad - \quad - \quad = r. \end{array} \qquad (27)$$

We shall now show that the set of relations R_2 (24) implies still another set R_2''. Let us subtract the square of the last equation of (24) from the product of the second and third and reduce the result; we get

$$LA_1^2+MA_2^2+NA_3^2+2PA_1A_2+2QA_1A_3+2RA_2A_3=L,$$

where L, M, etc., and A_1, A_2, etc., are the cofactors of l, m, etc., and a_1, a_2, etc., in the respective determinants

$$\begin{vmatrix} l & p & q \\ p & m & r \\ q & r & n \end{vmatrix} \text{ and } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}.$$

In like manner we deduce the entire set of relations R_2'' as follows:

$$R_2'' : \begin{array}{lcl} LA_1^2+MA_2^2+ & \text{- - - - -} & =L, \\ LB_1^2+MB_2^2+ & \text{- - - - -} & =M, \\ LC_1^2+ & \text{- - - - -} & =N, \\ LA_1B_1+ & \text{- - - - -} & =P, \\ LA_1C_1+ & \text{- - - - -} & =Q, \\ LB_1C_1+ & \text{- - - - -} & =R. \end{array} \qquad (27'')$$

If the six equations of R_2'' be multiplied respectively by a_1^2, b_1^2, etc., and added, we get as in Art. 195,

$$La_1^2+Mb_1^2+Nc_1^2+2Pa_1b_1+2Qa_1c_1+2Rb_1c_1=L.$$

In like manner we deduce the entire set of relations R_2''' as follows:

$$R_2''' : \begin{array}{lcl} La_1^2+Mb_1^2+Nc_1^2+ & \text{- - - -} & =L, \\ La_2^2+ & \text{- - - - - - -} & =M, \\ La_3^2+ & \text{- - - - - - -} & =N, \\ La_1a_2+ & \text{- - - - - - -} & =P, \\ La_1a_3+ & \text{- - - - - - -} & =Q, \\ La_2a_3+ & \text{- - - - - - -} & =R. \end{array} \qquad (27''')$$

Thus we see that if we have given a complete family of automorphic forms quadratic in the columns of M, we can deduce from it three other sets of quadratic relations. In fact if we have given any one of the four sets R_2, R_2', R_2'', or R_2''', we can deduce from it the other three. The last set of relations R_2''' is a complete family of automorphic forms quadratic in the rows of M. It follows from this that if we have given a complete family of automorphic forms quadratic in the rows of M, such as $l'a_1^2+m'b_1^2+$ etc.; these define a group $G_3(K')$ which leaves invariant a conic K' whose equation is

$$L'x^2+M'y^2+N'z^2+2P'xy+2Q'xz+2R'yz=0.$$

If l', m', etc., are chosen so L', M', equal to l, m, etc., the conic K' is the same as the conic K of Theorem 14.

THEOREM 15. There is only one variety of group, viz.: $G_3(K)$, defined by a set of relations quadratic in the elements of M; this group may be defined in two ways, either by a set of relations quadratic in the elements of the rows or columns of M.

212. *Groups Defined by Linear and Quadratic Relations.* A set of quadratic relations R_2 imposes five conditions on eight independent elements of M; a set of linear relations R_p or R_l imposes two conditions on M. Evidently a quadratic and a linear set may be simultaneously imposed on the elements of M and leave us still one independent parameter. Suppose that R_2 and R_p are imposed simultaneously on the elements of M; these will define a one-parameter group $G_1(AK)$ leaving invariant a point and a conic. Similarly R_2 and R_l define a group $G_1(lK)$ whose geometric invariant is a line and a conic.

If we have given R_2 and R_p, we can derive from these two sets another linear set of the kind R_l. Let the set R_p be as follows:

$$R_p: \quad \begin{aligned} l'a_1+m'b_1+n'c_1&=l', \\ l'a_2+m'b_2+n'c_2&=m', \\ l'a_3+m'b_3+n'c_3&=n', \end{aligned} \tag{22}$$

and let R_2 be given by equations (24). Multiply the first equation of R_p in turn by la_1, pa_2, qa_3; the second equation of R_p by pa_1, ma_2, ra_3, the third by qa_1, ra_2, na_3; add the nine products and reduce by means of R_2, we get

$$R_l: \quad \begin{aligned} (ll'+pm'+qn')\,a_1+(pl'+mm'+rn')\,a_2+(ql'+nm'+nn')a_3 \\ =ll'+pm'+qn', \end{aligned} \tag{32}$$

similarly we get

$$\begin{aligned} (ll'+pm'+qn')b_1+(pl'+mm'+rn')\,b_2+(ql'+rm'+nn')\,b_3&=(pl'+mm'+rn'), \\ (ll'+pm'+qn')c_1+(pl'+mm'+rn')\,c_2+(ql'+rm'+nn')\,c_3&=(ql'+rm'+nn'). \end{aligned}$$

The set of relations of the kind R_l secures the invariance of the line

$$(ll'+pm'+qn')\,x+(pl'+mm'+rn')\,y+(ql'+rm'+nn')\,z=0.$$

But this line is no other than the polar of the point (l', m', n') with respect to the conic,

$$lx^2+my^2+nz^2+2pxy+2qxz+2ryz=0.$$

In like manner the two sets R_2 and $R_{l'}$ imply a linear set of the kind R_p which secures the invariance of the pole of the line l' with respect to K. Hence the two groups $G_1(AK)$ and $G_1(lK)$ are of the same variety. The geometric invariant of the group is the triangle $(AA'A'')$ and the conic K, related to it as shown in Art. 204; the group is designated by $G_1(AA'A''K)$.

If the two sets of relations R_2 and R_p are so related to each other that the following condition exists,

$$ll'^2+mm'^2+nn'^2+2pl'm'+2ql'n'+2rm'n'=0,$$

then there are two independent parameters and the group defined is a two-parameter group. The invariant point is on the conic K and the invariant line touches K at the invariant point. The group is designated by $G_2(AlK)$.

213. *Groups Defined by Other Sets of Relations.* When we examine the complete cubic family of automorphic forms we see at once that the group defined by such a family is not a continuous group; for a set of relations R derived from a cubic family imposes nine conditions on eight independent elements of M and these conditions can be satisfied by only a finite number of sets of values of these elements. Hence in our study of continuous groups of plane collineations we can not make use of complete families of automorphic forms of degree higher than two.

We can not impose simultaneously two independent sets of quadratic relations upon the elements of M and thereby obtain a continuous group; for in such a case the number of conditions (ten) again exceeds the number of independent elements (eight) of M, and if a group is thus obtained it must be a finite group. (The identical collineation alone constitutes a finite group.)

It may be possible, however, to obtain a continuous group by imposing simultaneously upon the elements of M two sets of quadratic relations R_2 and R_2' so related that the two invariant conics have a special relation to each other. The only relations between the conics that we need to consider are those of contact; for if two invariant conics cut each other at four no three of which are collinear points, these four points are invariant and the only collineation leaving them invariant is the identical one. We must examine, however, all possible cases of contact of two conics.

All possible cases of contact of two conics have already been discussed in Art. 125 and their figures given in Fig. 16. If two conics K and K' have contact of the first order, this reduces by one the number of conditions on the elements of M; but we still have nine relations on eight parameters and no continuous group. If K and K' have second order contact, there are eight relations on eight parameters and no continuous group. But if K and K' have a double contact at two points A' and A'', it is possible that we may have a one-parameter group. For among the ∞^2 collineations leaving the triangle $(AA'A'')$ invariant there are ∞^1 that also leave K invariant; these may also leave K' invariant at the same time. That this is the case will be shown in the next §, and the group obtained is no other than $G_1(AA'A''K)$. If K and K' have third order contact, there is no continuous group (of type I) leaving them simultaneously invariant. It is possible, however, to find a system of collineations which interchange K and K'; this case will be considered later.

214. *The Limiting Case* $\Delta(l) = 0$. In order to make the above discussion complete two limiting cases remain to be examined, viz.: the group defined by three sets of linear relations when the determinant of the three sets of constants Art. 207 vanishes, and the group defined by a set of quadratic

relations when the determinant of the invariant conic Art. 210 vanishes.

If we have three sets of linear relations R_p, $R_{p'}$ and $R_{p''}$ so related that the determinant

$$\Delta(l) \equiv \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 0,$$

then these three sets of numbers are linearly related. Hence there exist three relations, as follows:

$$\begin{aligned} \lambda l + \mu l' &= l'', \\ \lambda m + \mu m' &= m'', \\ \lambda n + \mu n' &= n''. \end{aligned}$$

$R_{p''}$ may therefore be replaced by $R_{\lambda p + \mu p'}$. But the determinant of R_p, $R_{p'}$ and $R_{\lambda p + \mu p'}$ vanishes for all values of λ and μ. Hence if T leaves invariant three points which lie on a line l, all points on l are also invariant. In this case the three equations

$$\begin{aligned} x &= a_1 x + b_1 y + c_1 z, \\ y &= a_2 x + b_2 y + c_2 z, \\ z &= a_3 x + b_3 y + c_3 z, \end{aligned}$$

have an infinite number of linearly related solutions, and hence their determinant,

$$\Delta(1) \equiv \begin{vmatrix} a_1 - 1 & b_1 & c_1 \\ a_2 & b_2 - 1 & c_2 \\ a_3 & b_3 & c_3 - 1 \end{vmatrix} = 0, \tag{33}$$

is of rank *1*, *i. e.*, its first minors all vanish. But this is just the condition Art. 112, that the collineations be perspectives. Therefore the group defined by R_p, $R_{p'}$, $R_{p''}$ and $\Delta(l) =$ i0 . a group in which the collineations are all perspective collineations. It is a three-paramenter group, since the condition $\Delta(l) = 0$ decreases by one the number of independent conditions on the elements of M, and its geometric invariant is a line of invariant points.

If we have given three linear relations R_l, $R_{l'}$ and $R_{l''}$ with the condition that the determinant

$$\Delta(l) = \begin{vmatrix} l & m & n \\ l' & m' & n' \\ l'' & m'' & n'' \end{vmatrix} = 0,$$

it may be shown in the same way that the group thus defined is a three-parameter group of perspective collineations whose geometric invariant is a point A and all lines through it.

We might now go on and determine all groups of perspective collineations defined by sets of linear relations on the elements of M. But this problem will be taken up later, and the results reached by other methods are only what the present method would yield.

215. *The Limiting Case $D'=0$.* In discussing the group $G(K)$ defined by R, we assumed Art. 210, that the determinant

$$D' \equiv \begin{vmatrix} l & p & q \\ p & m & r \\ q & r & n \end{vmatrix} \neq 0.$$

If $D'=0$, the quadratic function breaks into factors; thus

$$lx^2+my^2+nz^2+2pxy+2qxz+2ryz \equiv (\lambda x+\mu y+\nu z)\ (\lambda' x+\mu' y+\nu' z);$$

whence we have $l=\lambda\lambda'$, $m=\mu\mu'$, $n=\nu\nu'$, $p=\lambda\mu'=\lambda'\mu$, $q=\lambda\nu'=\lambda'\nu$, $r=\mu\nu'=\mu'\nu$. The six relations of (24) reduce to

$$\begin{aligned}
&(\lambda a_1+\mu a_2+\nu a_3)\ (\lambda' a_1+\mu' a_2+\nu' a_3)=\lambda\lambda',\\
&(\lambda b_1+\mu b_2+\nu b_3)\ (\lambda' b_1+\mu' b_2+\nu' b_3)=\mu\mu',\\
&(\lambda c_1+\mu c_2+\nu c_3)\ (\lambda' c_1+\mu' c_2+\nu' c_3)=\nu\nu',\\
&(\lambda a_1+\mu a_2+\nu a_3)\ (\lambda' b_1+\mu' b_2+\nu' b_3)=\lambda\mu' \text{ or } \lambda'\mu,\\
&(\lambda a_1+\mu a_2+\nu a_3)\ (\lambda' c_1+\mu' c_2+\nu' c_3)=\lambda\nu' \text{ or } \lambda'\nu,\\
&(\lambda b_1+\mu b_2+\nu b_3)\ (\lambda' c_1+\mu' c_2+\nu' c_3)=\mu\nu' \text{ or } \mu'\nu.
\end{aligned} \tag{34}$$

These relations define a mixed group $mG_4(ll')$ which will be discussed in § 3 of Chapter IV. Equations (34) may be factored in two ways, as follows:

$$\begin{aligned}
\lambda a_1+\mu a_2+\nu a_3&=\lambda, \\
\lambda b_1+\mu b_2+\nu b_3&=\mu, \\
\lambda c_1+\mu c_2+\nu c_3&=\nu,
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
\lambda' a_1+\mu' a_2+\nu' a_3&=\lambda', \\
\lambda' b_1+\mu' b_2+\nu' b_3&=\mu', \\
\lambda' c_1+\mu' c_2+\nu' c_3&=\nu',
\end{aligned} \tag{34'}$$

or

$$\begin{aligned}
\lambda a_1+\mu a_2+\nu a_3&=\lambda', \\
\lambda b_1+\mu b_2+\nu b_3&=\mu', \\
\lambda c_1+\mu c_2+\nu c_3&=\nu',
\end{aligned}
\quad \text{and} \quad
\begin{aligned}
\lambda' a_1+\mu' a_2+\nu a'_3&=\lambda, \\
\lambda' b_1+\mu' b_2+\nu' b_3&=\mu, \\
\lambda' c_1+\mu' c_2+\nu' c_3&=\nu.
\end{aligned} \tag{34''}$$

Equations (34′) define a group $G_4(ll')$; this is already included in the list of Art. 209. Equations (34″) define a system of collineations which interchange the two lines $\lambda x+\mu y+\nu z=0$

and $\lambda' x+\mu' y+\nu' z=0$; this system of collineations does not form a group.

216. *List of Groups.* We have now found the entire list of subgroups of G_8 of type I which are defined by linear relations on the elements of M, or by quadratic relations, or by any combination of linear and quadratic relations. In addition to the list given in Art. 209, we have three more groups one set of whose defining relations is quadratic, viz.: $G_3(K)$, $G_2(AlK)$ and $G_1(AA'A''K)$. Our list is now in exact agreement with that given in Theorem 12. We may now restate Theorem 12 in this form:

THEOREM 16. The general projective group G_8 has only eleven varieties of continuous subgroups of type I which are defined by linear and quadratic relations on the elements of M; these are the groups enumerated in Theorem 12.

217. *The Condition* $\Delta(1)=0$. If we have given the family of automorphic forms linear in the rows of M,

$$\begin{aligned} la_1+mb_1+nc_1&=l,\\ la_2+mb_2+nc_2&=m,\\ la_3+mb_3+nc_3&=n, \end{aligned} \tag{22}$$

we can eliminate l, m, and n, and thus obtain the condition

$$\Delta(1)\equiv\begin{vmatrix} a_1-1 & b_1 & c_1 \\ a_2 & b_2-1 & c_2 \\ a_3 & b_3 & c_3-1 \end{vmatrix}=0. \tag{33}$$

The family linear in the columns of M gives the same condition. This necessary condition for a group defined by a linear set of relations is not a new independent condition on the elements of M. It is a sufficient condition for a group defined by a set of linear relations. For if the condition (33) be given in determinant form, there exists a set of numbers l, m, n, such that if the columns (or rows) be multiplied respectively by the numbers and the rows (or columns) added, each of the three sums will be zero. These sums give us at once the family of automorphic forms linear in the rows (or columns) of M.

We may also eliminate l, m, n, p, etc., from the quadratic family and obtain the condition,

$$\Delta_2(1) \equiv \begin{vmatrix} a_1^2-1 & b_1^2 & c_1^2 & 2a_1b_1 & 2a_1c_1 & 2b_1c_1 \\ a_2^2 & b_1^2-1 & - & - & - & - \\ a_3^2 & - & - & - & - & - \\ a_1a_2 & - & - & - & - & - \\ a_1a_3 & - & - & - & - & - \\ a_2a_3 & - & - & - & - & - \end{vmatrix} = 0. \qquad (35)$$

This condition factors into

$$(\Delta^2-1)\begin{vmatrix} a_1-1 & b_1 & c_1 \\ a_2 & b_2-1 & c_2 \\ a_3 & b_3 & c_3-1 \end{vmatrix} \cdot \begin{vmatrix} a_1+1 & b_1 & c_1 \\ a_2 & b_2+1 & c_2 \\ a_3 & b_3 & c_3+1 \end{vmatrix} \qquad (35')$$

$$\equiv (\Delta+1)(\Delta-1)\Delta(1)\Delta(-1)=0.$$

Here we must discard the first and last factors, since they come from the introduction of the transformations T, Art. 181, when the quadratic family is formed by the process of Art. 193. Since $\Delta-1$ always vanishes for the relations R_2, we cannot assert here that $\Delta(1)$ will also vanish. This question will be settled in § 6.

C. REDUCIBLE GROUPS AND CANONICAL FORMS OF GROUPS.

218. *Reducible Groups of Plane Collineations.* We shall now introduce the important conception of reducible groups of collineations. Groups of collineations may be divided into two distinct classes, reducible and irreducible. If a group G has the property that for each collineation in the group certain elements of its matrix not in the principal diagonal are always zero while the determinant of the matrix does not vanish, the group is said to be reducible. Furthermore every group G', equivalent to G according to the formula, $G'=S^{-1}GS$, where S is any collineation with non-vanishing determinant, is also said to be reducible.

Each group G' in the infinite system of equivalent reducible groups may by a suitable transformation of coordinates be brought into the form of G having certain zero elements in its matrix, which therefore may be called the reduced form of the reducible group. The general projective group G_8 is according to the definition not a reducible group. Every reducible group of plane collineations is therefore a subgroup of G_8. We wish to determine all varieties of such subgroups.

219. *Reduced Matrices of Subgroups of G_8.* The matrix of each of the subgroups of G_8 enumerated in Art. 209 may be written in a reduced form. For example, the group $G_6(A)$ is defined by the homogeneous and symmetrical relations

$$R_p : \begin{array}{l} la_1 + mb_1 + nc_1 = l, \\ la_2 + mb_2 + nc_2 = m, \\ la_3 + mb_3 + nc_3 = n. \end{array} \tag{22}$$

If we make $l = 1$, $m = 0$, $n = 0$, in R_p, these equations reduce to $a_1 = 1$, $a_2 = 0$, $a_3 = 0$. The matrix M reduces to

$$\begin{vmatrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{vmatrix},$$

and the invariant point (l, m, n) becomes $(1, 0, 0)$, one vertex of the triangle of reference. It is easily verified that if $a = 1$, $a_2 = 0$, $a_3 = 0$ and $\alpha_1 = 1$, $\alpha_2 = 0$, $\alpha_3 = 0$, then $A_1 = 1$, $A_2 = 0$, $A_3 = 0$. There are three equivalent reduced forms for the matrix of this group, viz.:

$$\left\|\begin{matrix} 1 & b_1 & c_1 \\ 0 & b_2 & c_2 \\ 0 & b_3 & c_3 \end{matrix}\right\|, \qquad \left\|\begin{matrix} a_1 & 0 & c_1 \\ a_2 & 1 & c_2 \\ a_3 & 0 & c_3 \end{matrix}\right\|, \qquad \left\|\begin{matrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{matrix}\right\|.$$

It may readily be verified by multiplication of matrices that each of these reduced matrices has both group properties, and represents therefore a reduced group.

In like manner if we make $l = 1$, $m = 0$, $n = 0$ in the relations R_l which define $G_6(l)$, we get $a_1 = 1$, $b_1 = 0$, $c_1 = 0$. The matrix M reduces to

$$\begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

and the invariant line becomes the line, $x = 0$, of the triangle of reference. Without going further into details we may write down at once one reduced form of the matrix of each of the other subgroups of G_8 defined by linear relations.

$$G_5(Al) \equiv \begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}, \qquad G_4(A, l'') \equiv \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ 0 & 0 & 1 \end{vmatrix},$$

$$G_4(AA') \equiv \begin{vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & 0 \\ a_3 & 0 & c_3 \end{vmatrix}, \qquad G_4(ll') \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & 0 \\ a_3 & b_3 & c_2 \end{vmatrix},$$

$$G_3(AA'l') \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & 0 \\ a_3 & 0 & c_3 \end{vmatrix}, \qquad G_2(AA'A'') \equiv \begin{vmatrix} 1 & 0 & 0 \\ 0 & b_2 & 0 \\ 0 & 0 & c_3 \end{vmatrix}.$$

It is easy to show by matrix multiplication that both group properties hold for each of these matrices, and that each of these eight subgroups of G_8 defined by linear relations is a reduced group.

220. *Necessary Condition of Reducibility.* We have thus far shown that every group whose defining equations are linear is a reducible group. We wish now to find a necessary condition of reducibility. A reducible group G' is one that may be transformed by a collineation S, thus $SG'S^{-1} = G$, such that for each collineation in G certain elements of its matrix are always zero, this being its reduced form.

Let us first assume that one of the elements in the principal diagonal of M, say a_1, is always zero. Putting $a_1 = 0$ and $\alpha_1 = 0$ in equations (10), Art. 172, we have $A_1 = a_2\beta_1 + a_3\gamma_1$, which is not zero; and we have no reducible group. A_1 will vanish if we assume say $a_2 = 0$ and $\gamma_1 = 0$; but then A_2 and C_1 will not vanish. A_1 will vanish if we assume $a_2 = a_3 = 0$, or $\beta_1 = \gamma_1 = 0$; but then the determinant of M vanishes and we have only pseudo-collineations. Hence an element in the principal diagonal of M can not be zero for all collineations in a group.

Next let us assume that some element not in the principal diagonal is zero; by suitable interchanges of rows and columns this element may be brought into the upper right hand corner of M; hence without loss of generality we may assume c_1 is always zero. Putting $c_1 = \gamma_1 = 0$ in (10) we get $C_1 = c_2\beta_1$, and we have no reducible group unless $c_2 = 0$ or $\beta_1 = 0$. Making $c_1 = c_2 = 0$ and $\gamma_1 = \gamma_2 = 0$, we get $C_1 = C_2 = 0$, and have a reducible group; making $c_1 = b_1 = 0$ and $\gamma_1 = \beta_1 = 0$ we get

$c_1 = b_1 = 0$ and we have another reducible group. Each of these two matrices (or their equivalents) viz.:

$$M_1 \equiv \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & c_3 \end{vmatrix}, \text{ and } M_2 \equiv \begin{vmatrix} a_1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix},$$

gives us the largest possible reducible group in G_8. Every other reducible group in G_8 will be contained within one of these. Hence in seeking for a general criterion of reducibility we need only consider these two. In the first case c_3 and in the second a_1 may be made unity without loss of generality.

The matrix M_2 shows at once that the line $x = 0$ is transformed into itself by every collineation of the group G_l; hence the transform of G_l, viz.: $G_l' = S^{-1} G_l S$, is a group which transforms into itself some linear function as $l'x + m'y + n'z$. But the necessary and sufficient condition for the invariance of this linear function under T is the set of linear relations

$$\begin{aligned} l'a_1 + m'a_2 + n'a_3 &= l', \\ l'b_1 + m'b_2 + n'b_3 &= m', \\ l'c_1 + m'c_2 + n'c_3 &= n'. \end{aligned} \tag{23}$$

The matrix M_1 shows that the point $(0, 0, 1)$ is transformed into itself by every collineation of the group G_p; hence the transform of G_p, viz. G'_p, leaves invariant some point whose coordinates are (l, m, n). The necessary and sufficient condition for the invariance under T of the ratios of the three numbers l, m, n is the set of linear relations

$$\begin{aligned} la_1 + mb_1 + nc_1 &= l, \\ la_2 + mb_2 + nc_2 &= m, \\ la_3 + mb_3 + nc_3 &= n. \end{aligned} \tag{22}$$

Hence for these two reducible groups G_p' and G_l', and therefore for all reducible groups with a smaller number of parameters, a necessary condition of reducibility is the existence of a set of linear relations in the elements of the rows or columns of M. We have now established the following:

THEOREM 17. A necessary and sufficient condition that a subgroup of G_8 be reducible is that at least one set of its defining relations, R, be linear in the elements of M.

It remains to be pointed out that the groups $G_2(AlK)$ and $G_1(AA'A''K)$ are each reducible and that $G_3(K)$ is irreducible. We have thus far found two irreducible groups, viz: G_8 and $G_3(K)$. It will be shown later that these are the only irreducible groups of plane collineations.

221. *The Orthogonal Group.* The group $G_3(K)$ is irreducible and hence its matrix can not be brought to a canonical form containing two or more zeros; but nevertheless its defining equations R_2 may be reduced to certain canonical forms. These we now proceed to find.

The group $G_3(K)$ leaves invariant in form the quadratic function

$$f \equiv lx^2+my^2+nz^2+2pxy+2qxz+2ryz.$$

From the theory of quadratic forms we know that f may be brought by linear transformation to the form

$$x^2+y^2+z^2.$$

In this case the triangle of reference is one of the self-polar triangles of the invariant conic $f=0$. We may, therefore, without loss of generality set $l=m=n=1$ and $p=q=r=0$ in the relations R_2, equations (24). The relations R_2 reduce to

$$\begin{array}{ll} a_1^2+a_2^2+a_3^2=1, & a_1b_1+a_2b_2+a_3b_3=0, \\ b_1^2+b_2^2+b_3^2=1, & a_1c_1+a_2c_2+a_3c_3=0, \\ c_1^2+c_2^2+c_3^2=1, & b_1c_1+b_2c_2+b_3c_3=0. \end{array} \tag{36}$$

These are the well-known relations on the elements of M which define the orthogonal group in three variables.

For the same values of the constants l, m, etc., equations (27) reduce to

$$\begin{array}{ll} A_1^2+B_1^2+C_1^2=1, & A_1A_2+B_1B_2+C_1C_2=0, \\ A_2^2+B_2^2+C_2^2=1, & A_1A_3+B_1B_3+C_1C_3=0, \\ A_3^2+B_3^2+C_3^2=1, & A_2A_3+B_2B_3+C_2C_3=0; \end{array} \tag{36'}$$

equations (27″) reduce to

$$\begin{array}{ll} A_1^2+A_2^2+A_3^2=1, & A_1B_1+A_2B_2+A_3B_3=0, \\ B_1^2+B_2^2+B_3^2=1, & A_1C_1+A_2C_2+A_3C_3=0, \\ C_1^2+C_2^2+C_3^2=1, & B_1C_1+B_2C_2+B_3C_3=0; \end{array} \tag{36''}$$

and (27‴) become

$$\begin{array}{ll} a_1^2+b_1^2+c_1^2=1, & a_1a_2+b_1b_2+c_1c_2=0, \\ a_2^2+b_2^2+c_2^2=1, & a_1a_3+b_1b_3+c_1c_3=0, \\ a_3^2+b_3^2+c_3^2=1, & a_2a_3+b_2b_3+c_2c_3=0. \end{array} \tag{36'''}$$

These four sets of relations hold for the orthogonal group; if we have given any one set of these relations we can deduce the other three.

THEOREM 18. The orthogonal group is a special case of the group $G_3(K)$ when the equation of the conic K is in the canonical form

$$x^2+y^2+z^2=0.$$

222. *The Second Canonical Form of* $G_3(K)$. The quadratic function f can also be brought by linear transformation to the canonical form $f\equiv y^2-xz$. We may therefore set $l=n=p=r=0$ and $m=-q=1$. The relations R_2 (24) then reduce to

$$a_2^2=a_1a_3, \qquad b_2^2=b_1b_3, \qquad c_2^2=c_1c_3, \qquad 2a_2b_2=a_1b_3+a_3b_1,$$
$$2a_2c_2=a_1c_3+a_3c_1-1, \qquad 2b_2c_2=b_1c_3+b_3c_1.$$

Let us put $a_1=\alpha^2$, $a_3=\gamma^2$, $c_1=\beta^2$, $c_3=\delta^2$; we then find $b_1=2\alpha\beta$, $b_3=2\gamma\delta$, $b_2=\alpha\delta+\beta\gamma$, and $(\alpha\delta-\beta\gamma)=1$. The matrix M is now in terms of α, β, γ, δ, thus

$$M\equiv\begin{Vmatrix} \alpha^2 & 2\alpha\beta & \beta^2 \\ \alpha\gamma & \alpha\delta+\beta\gamma & \beta\delta \\ \gamma^2 & 2\gamma\delta & \delta^2 \end{Vmatrix};$$

and the determinant of M is $(\alpha\beta-\gamma\delta)^3=1$. The invariant conic has the equation $y^2=xz$ and the invariant triangle of T coincides with the triangle of reference.

This form of T is connected in an interesting way with the one-dimensional transformation

$$x_1'=\alpha x'+\beta y'$$
$$y_1'=\gamma x'+\delta y'$$

Taking the square of the first, the product of the first and second and the square of the second, we get after replacing x'^2 by x, y'^2 by z, and $x'y'$ by y,

$$\begin{aligned} x_1&=\alpha^2x+2\alpha\beta y+\beta^2z, \\ y_1&=\alpha\gamma x+(\alpha\delta+\beta\gamma)y+\beta\delta z, \\ z_1&=\gamma^2x+2\gamma\delta y+\delta^2z, \end{aligned} \qquad (37)$$

with the condition $y_1^2-x_1z_1=y^2-xz$.

This shows that the conic $y^2=xz$ is invariant and that the transformations in the group $G_3(K)$ have a one-to-one correspondence with those of the group G_3 of one-dimensional projective transformations. When two groups of transformations are so related to each other that there is a one-to-one correspondence between the transformations of the two groups, the resultant of any two transformations corresponding to the resultant of their correspondents, then they break up into subgroups in exactly the same way and have exactly the same structure. Two such groups are said to be holoedrically isomorphic.

THEOREM 19. The group G_3 of one-dimensional projective transformations and $G_3(K)$ are holoedrically isomorphic.

§ 5. Groups of Other Types Defined by Linear Relations.

The groups determined in the last § are all of type I. It was assumed that the elements of the matrix M were not subject to any of the conditions, Art. 113 117, that cause its characteristic equation to have multiple roots or the first minors to vanish, *i. e.* the collineations were assumed not to be of any of the secondary types. We shall now take up each type separately and determine the conditions under which all the collineations of a given group belong to a given type.

TYPE II.

223. *No Six- or Seven-Parameter Groups of Type II.* The condition that a collineation shall be of type II is, Art. 114, that its characteristic equation,

$$\begin{vmatrix} a_1-\rho & b_1 & c_1 \\ a_2 & b_2-\rho & c_2 \\ a_3 & b_3 & c_3-\rho \end{vmatrix} = 0,$$

shall have a double root; *i. e.*, that its discriminant shall vanish. The vanishing of the discriminant D lays one condition on the elements of M. There are therefore ∞^7 collineations of type II. These do not form a seven-parameter group because this one condition does not comply with the necessary and sufficient conditions laid down in Theorem 11.

The largest group that could exist is a six-parameter group defined by one set of linear relations R_p or R_l. If we lay upon the elements of M simultaneously the conditions R_p and D, or R_l and D, we have only ∞^5 collineations. There can not be, therefore, a six-parameter group of type II.

There may or may not be a five-parameter group of type II, so far as we yet know. We must examine the system of ∞^5 collineations defined by say R_p and D. If the resultant of any two collineations of this system is also of type II, then this system has the first group property.

224. *No Five-Parameter Group of Type II.* The group of collineations defined by R_p alone is reducible; its matrix and its characteristic equation may therefore be written in the reduced forms:

$$M \equiv \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{vmatrix}, \quad \text{and } \Delta(\rho) \equiv \begin{vmatrix} a_1-\rho & b_1 & 0 \\ a_2 & b_2-\rho & 0 \\ a_3 & b_3 & 1-\rho \end{vmatrix} = 0. \qquad (38)$$

One root of $\Delta(\rho)$ is given by the factor $\rho - 1 = 0$; the other two roots of $\Delta(\rho)$ are the roots of the quadratic

$$\rho^2 - (a_1 + b_2)\rho + (a_1 b_2 - a_2 b_1) = 0. \qquad (39)$$

Since $\Delta(\rho) = 0$ is to have a double root, two cases arise; either 1 is one root of (39) or (39) has equal roots. We must consider these cases separately.

(1). Let us suppose first that 1 is one root of (39); then we have for T the condition

$$1 - (a_1 + b_2) + (a_1 b_2 - a_2 b_1) = 0. \qquad \text{(i)}$$

The same condition holds also for T_1, the elements of whose matrix are α_1, β_1, etc.; thus:

$$1 - (\alpha_1 + \beta_2) + \alpha_1\beta_2 - \alpha_2\beta_1 = 0. \qquad \text{(ii)}$$

We wish to see if the same condition holds also for T_2, where $TT_1 = T_2$, *i. e.*, is it true that the vanishing of (i) and (ii) makes the function

$$1 - (A_1 + B_2) + A_1 B_2 - A_2 B_1 \qquad \text{(iii)}$$

also vanish?

Since the determinant of T_2 is equal to the product of the determinants of T and T_1, we have

$$A_1 B_2 - A_2 B_1 \begin{aligned} &= (a_1 b_2 - a_2 b_1)\ (\alpha_1\beta_2 - \alpha_2\beta_1), \\ &= (a_1 + b_2 - 1)\ (\alpha_1 + \beta_2 - 1). \end{aligned}$$

The function (iii) becomes

$$1-(A_1+B_2)+(a_1+b_2-1)(\alpha_1+\beta_2-1).$$

Substituting for A_1, B_2, their values from equation (10) Art. 172, after putting $c_1=c_2=0$, $\gamma_1=\gamma_2=0$, and $c_3=\gamma_3=1$, we get

$$(a_1-1)(\beta_2-1)+(b_2-1)(\alpha_1-1)-(a_2\beta_1+b_1\alpha_2). \qquad \text{(iiii)}$$

This quantity can not vanish because of the presence of the terms $a_2\beta_1$ and $b_1\alpha_2$, which do not occur in (i) or (ii). Hence T_2 is not of type II.

(2). Let us next suppose that 1 is not a double root of $\Delta(\rho)=0$, but that (39) has equal roots. The condition that (39) has equal roots is

$$4(a_1b_2-a_2b_1)-(a_1+b_2)^2=0. \qquad \text{(j)}$$

Writing down the same conditions for T_1 and T_2 we have the problem: Do the first two conditions

$$4(a_1b_2-a_2b_1)-(a_1+b_2)^2=0, \qquad \text{(j)}$$

$$4(\alpha_1\beta_2-\alpha_2\beta_1)-(\alpha_1+\beta_2)^2=0, \qquad \text{(jj)}$$

$$4(A_1B_2-A_2B_1)-(A_1+B_2)^2 \qquad \text{(jjj)}$$

cause the third expression to vanish? Since $\Delta_2=\Delta\Delta_1$, we may write the third in the form

$$(a_1+b_2)^2(\alpha_1+\beta_2)^2-4(a_1\alpha_1+a_2\beta_1+b_1\alpha_2+b_2\beta_2)^2. \qquad \text{(jjjj)}$$

This function does not vanish because of the presence of the two terms $a_2\beta_1$ and $b_1\alpha_2$; hence again T_2 is not of type II.

Both cases (1) and (2) lead to the same result, viz.: the conditions R_2 and D do not define a five-parameter group of type II.

In like manner we may take the two conditions R_1 and D and write M and $\Delta(\rho)$ in the reduced forms, thus:

$$M\equiv\begin{Vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{Vmatrix} \text{ and } \Delta(\rho)\equiv\begin{vmatrix} a_1-\rho & b_1 & c_1 \\ a_2 & b_2-\rho & c_2 \\ 0 & 0 & 1-\rho \end{vmatrix}=0. \qquad (38')$$

The problem is now identical with the one just solved for R_2 and D and leads to the same results.

We have thus proved that there is no five-parameter group of type II defined by D and a set of linear relations on M. It follows that there can be no five-parameter group of type II defined in any way. For if it should be found that D and two or more sets of linear relations or D and R_2 define a group, the number of parameters would have to be less than five.

THEOREM 20. There are no five- six- or seven-parameter groups of type II.

225. *Four-Parameter Groups of Type II.* We shall now prove the existence of four-parameter groups of type II. We saw in case (1) above that conditions (i) and (ii) were not sufficient to cause (iii) to vanish because of the presence of the terms $a_2\beta_1$ and $b_1\alpha_2$ in (iii) not involved in (i) and (ii). Hence a necessary condition for the vanishing of (iiii) is either $b_1=0$ and $\beta_1=0$, or $a_2=0$ and $\alpha_2=0$. Each one of these conditions is also sufficient, for on either supposition we have, since 1 is a root of (39), $a_1=1$ and $\alpha_1=1$, or $b_2=1$ and $\beta_2=1$; and either supposition causes (iiii) to vanish.

We found also in case (2) that a necessary condition for the vanishing of (jjjj) is either $b_1=0$ and $\beta_1=0$, or $a_2=0$ and $\alpha_2=0$. Each of these conditions is also sufficient, for on either supposition we have from (j) $b_2=a_1$ and $\beta_2=\alpha_1$ and these cause (jjjj) to vanish.

We found also in case (2) that a necessary condition for the vanishing of (jjjj) is either $b_1=0$ and $\beta_1=0$, or $a_2=0$ and $\alpha_2=0$. Each of these conditions is also sufficient, for on either supposition we have from (j) $b_2=a_1$ and $\beta_2=\alpha_1$ and these cause (jjjj) to vanish.

Thus cases (1) and (2) lead to the same result and we have a four-parameter group of type II, if $b_1=0$ or $a_2=0$. Taking all combinations we have four cases to examine, viz.: $b_1=0$ with 1 a single or a double root of $\Delta(\rho)=0$, and $a_2=0$ with 1 a single or a double root. The proof holds in all four cases. For these four cases the matrix M reduces respectively to

$$(1)\ \begin{Vmatrix} a_1 & 0 & 0 \\ a_2 & a_1 & 0 \\ a_3 & b_3 & 1 \end{Vmatrix},\ (2)\ \begin{Vmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & b_3 & 1 \end{Vmatrix},\ (3)\ \begin{Vmatrix} a_1 & b_1 & 0 \\ 0 & a_1 & 0 \\ a_3 & b_3 & 1 \end{Vmatrix},\ (4)\ \begin{Vmatrix} a_1 & b_1 & 0 \\ 0 & 1 & 0 \\ a_3 & b_3 & 1 \end{Vmatrix}.$$

Let the invariant figure of type II, Fig. (14 II), be lettered

A, A', l, l', where A and l are respectively the point and line corresponding to the single root of $\Delta(\rho)=0$ and A' and l' those corresponding to the double root of $\Delta(\rho)=0$. The invariant lineal element of matrix (1) is Al', of (2) is $A'l$, of (3) is Al', of (4) is $A'l'$. Since (1) and (3) are the same we have three distinct cases and hence three distinct four-parameter groups, one for each distinct lineal element in the figure $AA'll'$. These groups are designated respectively by $G_4'(Al')$, $G_4'(A'l)$, $G_4'(A'l')$.

Since the vanishing of either b_1 or a_2 is both necessary and sufficient to establish the first group property we have proved the following important theorem :

THEOREM 21. **A necessary and sufficient condition for the existence of a group of collineations of type II is that the invariant figures of all the collineations in the system have in common the same lineal element.**

226. *Other Groups of Type II.* Each of the above four-parameter groups in its unreduced form is defined by the following sets of relations : R_p, R_l, $S\lambda l$ and D. In precisely the same manner we can show the existence of groups of type II as follows : R_p, R_p' and D; R_l, R_l' and D; R_p, $R_{p'}$, R_l, $S\lambda l$ and D, where the relation $S\lambda l$ exists between the coordinates of the single line and double point. These groups are designated respectively by $G_3'(AA')$, $G_3'(ll')$, $G_2(AA'l')$. We have therefore six varieties of groups of type II defined by linear relations on the elements of M.

In order to make the discussion complete we should examine for the group property the set of ∞^2 collineations defined by R_2 and D. But since the group defined by R_2 alone is irreducible, the application of the above process would be very tedious. Later we shall attack the problem by an indirect method and reach a negative result.

THEOREM 22. **There are six varieties of groups of type II defined by linear relations on the elements of M, viz.:**
$G_4'(Al')$, $G_4'(A'l)$, $G_4'(A'l')$, $G_3'(AA')$, $G_3'(ll')$, $G_2'(AA'l')$.

TYPE III.

We shall now investigate the conditions that must be satisfied by collineations of type III, if they are to form a group. The necessary and sufficient condition that a collineation be of type III is that its characteristic equation $\Delta(\rho) = 0$ shall have three equal roots, for which its first minors do not all vanish. This is equivalent to two conditions on the elements of M, call them D and D', and shows that there are ∞^6 collineations of type III. For the same reasons that hold in the case of type II, we see that there can be no five- or six-parameter groups of type III.

227. *The Group* $G_3''(Al)$. We shall first investigate the existence of a four-parameter group of type III, defined by R_p, D and D'. We may write the matrix M and the characteristic equation $\Delta(\rho)$ in the same form as in Art. 224. The factor $\rho - 1$ gives one root of $\Delta(\rho) = 0$ which must be a triple root. It follows that the quadratic equation (39) must be satisfied by 1 and also have equal roots. This is equivalent to saying that the two conditions, which gave us cases (1) and (2) of Art. 224, hold simultaneously. Since these two conditions separately lead to the same result, when combined they give us that same result. Hence we infer that there is no four-parameter group of type III defined by R_p, D and D', or by R_l, D and D'. It also follows that a necessary and sufficient condition for a group of type III is the vanishing of either b_1 or a_2 in the reduced form of the matrix M.

The matrix M of this group may therefore be written in either form

$$(1) \left\| \begin{matrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & b_3 & 1 \end{matrix} \right\| \text{ or } (2) \left\| \begin{matrix} 1 & b_1 & 0 \\ 0 & 1 & 0 \\ a_3 & b_3 & 1 \end{matrix} \right\| .$$

The group leaves invariant the lineal element Al, *i. e.*, the collineations of the group all have the same invariant figure; it is designated by $G_3''(Al)$. We have thus proved that if a system of collineations of type III form a group, it is necessary and sufficient that each collineation of the system leave invariant the same lineal element. It also follows that $G_3''(Al)$ is the only group of type III defined by linear relations on the elements of M.

THEOREM 23. A necessary and sufficient condition for the existence of a group of collineations of type III is that the invariant figures of all the collineations in the system have in common the same lineal element. There is but one group of type III defined by linear relations on the elements of M.

TYPE IV.

In order that a collineation be of type IV it is necessary and sufficient that its characteristic equation $\Delta(\rho)=0$ have a double root for which all the first minors of $\Delta(\rho)$ vanish. This is equivalent to three non-linear relations D, D', D'' on the elements of M. It follows that there are ∞^5 collineations of type IV and no four- or five-parameter groups of this type.

228. *Three-parameter Groups of Type IV.* Let us examine for the first group property the system of ∞^5 collineations defined by R_p, D, D', D''. We may take M and $\Delta(\rho)$ as before in the reduced forms:

$$M \equiv \begin{vmatrix} a_1 & b_1 & 0 \\ a_2 & b_2 & 0 \\ a_3 & b_3 & 1 \end{vmatrix} \text{ and } \Delta(\rho) \equiv \begin{vmatrix} a_1-\rho & b_1 & 0 \\ a_2 & b_2-\rho & 0 \\ a_3 & b_3 & 1-\rho \end{vmatrix} = 0. \qquad (38)$$

One root of $\Delta(\rho)=0$ is 1 and this value makes six of the first minors of $\Delta(\rho)$ vanish identically. It must make the remaining three vanish also, viz.:

$$\begin{vmatrix} a_1-\rho & b_1 \\ a_2 & b_2-\rho \end{vmatrix}, \quad \begin{vmatrix} a_1-\rho & b_1 \\ a_3 & b_3 \end{vmatrix}, \quad \begin{vmatrix} a_2 & b_2-\rho \\ a_3 & b_3 \end{vmatrix}.$$

This gives us three conditions as follows:

$$\begin{aligned} &(a_1-1)\ (b_2-1)=a_2 b_1, \\ &(a_1-1)\ b_3=a_3 b_1, \\ &(b_2-1)\ a_3=a_2 b_3. \end{aligned} \qquad (40)$$

These three are not independent, since the product of the second and third gives the first.

Applying the first condition exactly as in case (1) Art. 224 we find that a necessary and sufficient condition for a group is either $b_1=0$ or $a_2=0$ in M. If $b_1=0$, a_1-1 can not vanish without $\Delta(\rho)$ vanishing identically. Substituting $b_1=0$ in (40) we get the conditions $b_1=0$, $b_2=1$, $b_3=0$. In like manner if $a_2=0$, we have the conditions $a_1=1$, $a_2=0, a_3=0$. The matrix M may now be written in either of the equivalent forms:

$$\begin{Vmatrix} a_1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{Vmatrix} \text{ or } \begin{Vmatrix} 1 & b_1 & 0 \\ 0 & b_2 & 0 \\ 0 & b_3 & 1 \end{Vmatrix}.$$

These show that all collineations of the group leave invariant the line corresponding to the single root of $\Delta(\rho)$ and the points on it corresponding to the double root; *i. e.*, all points on the line. The group may therefore be designated by $H_3(l)$.

In exactly the same manner it may be shown that the system of collineations defined by R_l, D, D', D'' form a three-parameter group whose matrix may be written in the equivalent reduced forms:

$$\begin{Vmatrix} 1 & 0 & 0 \\ a_2 & b_2 & c_2 \\ 0 & 0 & 1 \end{Vmatrix} \text{ or } \begin{Vmatrix} a_1 & b_1 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

These forms show that the collineations of the group all leave invariant the point corresponding to the single root of $\Delta(\rho)$ and the lines through it corresponding to the double root; *i. e.*, all lines through it. This group will be designated by $H_3(A)$.

THEOREM 24. A necessary and sufficient condition for the existence of a group of collineations of type IV is that the invariant figures of all collineations of the system have in common either the same vertex or the same axis.

Other Groups of Type IV. In a similar manner we can prove the existence of a two-parameter group $H_2(AA')$ in which all the collineations of the group have a common vertex A, and axes intersecting in a common point A'; also a two-parameter group $H_2(ll')$ in which all the collineations have a common axis l, and vertices on a common line l'; also a one-parameter group $H_1(A, l)$, in which all the collineations have a common vertex and a common axis.

THEOREM 25. There are five varieties of groups of type IV defined by linear relations on the elements of M, viz.: $H_3(A)$, $H_3(l)$, $H_2(AA)$, $H_2(ll')$, $H_1(A,l)$.

TYPE V.

229. The corresponding results for type V may be readily deduced from those for type IV. There is one added condition, viz.: That $\Delta(\rho)=0$ have a triple root instead of a double root and a single root. We infer at once that there are ∞^4 collineations of type V and no three- or four-parameter groups of type V.

If to the analysis of Art. 228 we add the condition that 1 is a triple root of $\Delta(\rho)=0$, we reach the conclusion that there is a two-parameter group of type V whose matrix may be written in either form,

$$\begin{Vmatrix} 1 & 0 & 0 \\ a_2 & 1 & 0 \\ a_3 & 0 & 1 \end{Vmatrix} \text{ or } \begin{Vmatrix} 1 & b_1 & 0 \\ 0 & 1 & 0 \\ 0 & b_3 & 1 \end{Vmatrix};$$

also another two-parameter group whose matrix may be either

$$\begin{Vmatrix} 1 & 0 & 0 \\ a_2 & 1 & c_2 \\ 0 & 0 & 1 \end{Vmatrix} \text{ or } \begin{Vmatrix} 1 & b_1 & c_1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{Vmatrix}.$$

All collineations belonging to the first group have in common the line of invariant points; all belonging to the second group have in common the pencil of invariant lines. We designate the first group by $H_2'(l)$ and the second by $H_2'(A)$. We also find a one-parameter group $H_1'(Al)$ in which all the collineations have in common the same axis and the same vertex. We also see that theorem 24 holds word for word for type V as well as for type IV.

THEOREM 26. There are three varieties of groups of type V defined by linear relations on the elements of M, viz.: $H_2'(A)$, $H_2'(l)$, $H_1'(Al)$.

§6. Normal Forms of Groups of Type I.

We shall now return to the theory of the groups of type I and continue by the use of the normal form of T the discussion of those groups begun in §4 by means of the homogeneous form of T. We shall express the relations R_p, R_l, R_k, etc., in the normal form of T and deduce therefrom many important results, the chief of which are what we shall call the k-relations.

The normal form of T will be taken with the proportionality factor ρ equal to unity, thus:

$$x_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}, \; etc.$$

Care must be taken as before to exclude from the discussion the form $\overline{T}$, viz.:

$$-x_1 = \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}, \; etc.$$

230. *Normal Form of R_p.* Using the normal form of the collineation T, page 104,, the linear relations R_p defining the group $G_6(A)$ become:

$$\begin{aligned}
l\begin{vmatrix} B & C & A \\ B' & C' & kA' \\ B'' & C'' & k'A'' \end{vmatrix} - m\begin{vmatrix} A & C & A \\ A' & C' & kA' \\ A'' & C'' & k'A'' \end{vmatrix} + n\begin{vmatrix} A & B & A \\ A' & B' & kA' \\ A'' & B'' & k'A'' \end{vmatrix} &\equiv \begin{vmatrix} l & m & n & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix} = l \\
l\begin{vmatrix} B & C & B \\ B' & C' & kB' \\ B'' & C'' & k'B'' \end{vmatrix} - m\begin{vmatrix} A & C & B \\ A' & C' & kB' \\ A'' & C'' & k'B'' \end{vmatrix} + n\begin{vmatrix} A & B & B \\ A' & B' & kB' \\ A'' & B'' & k'B'' \end{vmatrix} &\equiv \begin{vmatrix} l & m & n & 0 \\ A & B & C & B \\ A' & B' & C' & kB' \\ A'' & B'' & C'' & k'B'' \end{vmatrix} = m \qquad (41) \\
l\begin{vmatrix} B & C & C \\ B' & C' & kC' \\ B'' & C'' & k'C'' \end{vmatrix} - m\begin{vmatrix} A & C & C \\ A' & C' & kC' \\ A'' & C'' & k'C'' \end{vmatrix} + n\begin{vmatrix} A & B & C \\ A' & B' & kC' \\ A'' & B'' & k'C'' \end{vmatrix} &\equiv \begin{vmatrix} l & m & n & 0 \\ A & B & C & C \\ A' & B' & C' & C' \\ A'' & B'' & C'' & k'C'' \end{vmatrix} = n.
\end{aligned}$$

We may eliminate l, m, and n from these equations, as in Art. 217, and thus get:

$$\Delta(1) \equiv \begin{vmatrix} a_1 - 1 & b_1 & c_1 \\ |a_2| & |b_2| - 1 & |c_2| \\ |a_3| & |b_3| & |c_3| - 1 \end{vmatrix} = 0, \tag{42}$$

where $a_1|$ etc., are the elements of the matrix of the normal form of T. But this relation is no other than the characteristic equation of T, with ρ replaced by 1. Hence, by Art. 133, it breaks up into three factors as follows:

$$(\Delta' - 1)\ (k\Delta' - 1)\ (k'\Delta' - 1) = 0, \tag{42'}$$

where Δ' is the determinant of the invariant triangle of T. We wish to know the meaning of each of these factors.

The form of equations (41) shows that the point (l, m, n) is transformed into itself. But this can only be true for one of the invariant points of T. Suppose that it is the invariant point (A, B, C); substituting A, B, C for l, m, n in equations (41) we get $\Delta' A = A$, $\Delta' B = B$, $\Delta' C = C$. Hence we have $\Delta' = 1$. If the point (l, m, n) is (A', B', C'), this leads to the condition $k\Delta' = 1$; if (l, m, n) is (A'', B'', C''), then $k'\Delta' = 1$.

The conditions for the group $G_6(A)$ require that one factor of (42′) must vanish. If $\Delta' = 1$, the invariant triangles of the collineations in $G_6(A)$ all have the point (A, B, C) in common which is therefore the invariant point of the group. If $k\Delta' = 1$, (A', B', C') is the invariant point; if $k'\Delta' = 1$, (A'', B'', C'') is the invariant point.

231. *The k-Relation.* Suppose that the invariant point of $G_6(A)$ is the point (A, B, C). Since the relations R_p hold for T, T_1 and T_2 in $G_6(A)$, we have $\Delta' = 1$, $\Delta_1' = 1$, and $\Delta_2' = 1$. Equation X of Art. 179½ may be written in the form

$$k_2 k_2' \Delta_2'^3 = k k' \Delta'^3,\ k_1 k_1' \Delta_1'^3; \tag{X}$$

Substituting the values of the Δ''s in X we get

$$k_2 k_2' = k k'\ k_1 k_1'. \tag{43}$$

If the invariant point of $G_6(A)$ is (A', B', C'), we get in the same way the relation

$$\frac{k_2'}{k_2^2} = \frac{k' k_1'}{k^2 k_1^2}. \tag{43'}$$

If (A'', B'', C'') is the invariant point, we get

$$\frac{k_2}{k_2'^2} = \frac{k k_1}{k'^2 k_1'^2}. \tag{43''}$$

In all three cases we see that the product of the two cross-ratios in T_2 along the two sides of the invariant which meet in the common invariant point, both cross-ratios being taken in directions away from the point, is equal to the product of the corresponding cross-ratios of T and T_1.

232. This shows that we have a very simple relation among the k's of T, T_1 and T_2. The "k-relations" 43, 43′, 43″, are due to the fact that T, T_1, T_2, were written so as to have one root of their characteristic equations $= 1$. Since division of all coefficients a_1, b_1, etc., by a root of the characteristic equation allows us to throw *any* collineation T into this form, we may state:

THEOREM 27. The k-relations 43, 43′ and 43″ hold for the entire group G_8, provided its transformations be written as here directed.

233. *Normal Form of R_l in $G_6(l)$.* Using the normal form of T the relations R_l defining the group $G_6(l)$ become

$$\begin{aligned}
\lambda \begin{vmatrix} B & C & A \\ B' & C' & kA' \\ B'' & C'' & k'A'' \end{vmatrix} + \mu \begin{vmatrix} B & C & B \\ B' & C' & kB' \\ B'' & C'' & k'B'' \end{vmatrix} + \nu \begin{vmatrix} B & C & C \\ B' & C' & kC' \\ B'' & C'' & k'C'' \end{vmatrix} &= \lambda\,, \\
\lambda \begin{vmatrix} A & C & A \\ A' & C' & kA' \\ A'' & C'' & k'A'' \end{vmatrix} + \mu \begin{vmatrix} A & C & B \\ A' & C' & kB' \\ A'' & C'' & k'B'' \end{vmatrix} + \nu \begin{vmatrix} A & C & C \\ A' & C' & kC' \\ A'' & C'' & k'C'' \end{vmatrix} &= \mu\,, \\
\lambda \begin{vmatrix} A & B & A \\ A' & B' & kA' \\ A'' & B'' & k'A'' \end{vmatrix} + \mu \begin{vmatrix} A & B & B \\ A' & B' & kB' \\ A'' & B'' & k'B'' \end{vmatrix} + \nu \begin{vmatrix} A & B & C \\ A' & B' & kC'' \\ A'' & B'' & k'C'' \end{vmatrix} &= \nu\,.
\end{aligned} \tag{44}$$

Again we may eliminate λ, μ, ν from these equations and obtain, as in the case of R_p, the condition $\Delta(1) = 0$. We shall find as before the significance of each factor in (42′).

Multiplying the first equation by x, the second by y, and the third by z, and adding we get

$$\lambda\begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}+\mu\begin{vmatrix} x & y & z & 0 \\ A & B & C & B \\ A' & B' & C' & kA' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}+\nu\begin{vmatrix} x & y & z & 0 \\ A & B & C & C \\ A' & B' & C' & kC' \\ A'' & B'' & C'' & k'C'' \end{vmatrix}=\lambda x+\mu y+\nu z, \tag{45}$$

or

$$\begin{vmatrix} x & y & z & 0 \\ A & B & C & \lambda A+\mu B+\nu C \\ A' & B' & C' & k(\lambda A'+\mu B'+\nu C') \\ A'' & B'' & C'' & k'(\lambda A''+\mu B''+\nu C'') \end{vmatrix}=\lambda x+\mu y+\nu z. \tag{45'}$$

Equation (45) shows that the form of the function $\lambda x+\mu y+\nu z$ is invariant under the normal form of T. Its value is the same at a pair of corresponding points (x, y, z) and (x_1, y_1, z_1), and is therefore invariant under T at each of the invariant points of T. The function vanishes at all points of a certain line of the plane which can only be one of the sides of the invariant triangle of T.

234. *k-Relations in Subgroups of G_8.* The group $G_4(AA')$ is defined by two sets of relations R_p and $R_{p'}$. Suppose the invariant point determined by R_p is (A, B, C), then that determined by R_p must be either A' or A''; let us suppose it is A'. Since (A, B, C) is an invariant point of the group, we have the k-relation $k_2k_2'=kk'k_1k_1'$; since (A', B', C') is also invariant, we have $\frac{k_2'}{k_2^2}=\frac{k'k_1'}{k^2k_1^2}$. Combining these two relations we get

$$k_2=kk_1 \text{ and } k_2'=k'k_1'. \tag{46}$$

Thus for the group $G_4(AK)$ we have a double k-relation.

In like manner we can determine the k-relations in all the other groups of the list of Art. 209. The results are as follows: The groups $G_6(A)$, $G_6(l)$, and $G_4(Al)$ each have a single k-relation, viz., $k_2k_2'=kk'k_1k_1'$; the groups $G_5(Al)$, $G_4(AA')$, $G_4(ll')$, $G_3(AA'l)$, and $G_2(AA'A'')$ each have the double k-relations, viz., $k_2=kk_1$ and $k_2'=k'k_1'$.

It should be noted that each group whose invariant figure has at least one lineal element has the double k-relations.

THEOREM 28. Every subgroup of type I in G_8 which is defined by linear relations R on the elements of M, and whose invariant figure contains a lineal element has a double k-relation.

235. *The Normal Form of* $G_3(K)$. If we replace a_1, b_1, etc., in the relations R_2 by their values from the normal form of T, viz.:

$$a_1 \equiv \begin{vmatrix} B & C & A \\ B' & C' & kA' \\ B'' & C'' & k'A'' \end{vmatrix}, \quad b \equiv \begin{vmatrix} A & C & A \\ A' & C' & kA' \\ A'' & C'' & k'A'' \end{vmatrix}, \text{ etc.,}$$

we get the normal form of the conditions for the group $G_3(k)$. If these new relations be multiplied by x^2, y^2, etc., and added, we get

$$l \begin{vmatrix} x & y & z & 0 \\ A & B & C & A \\ A' & B' & C' & kA' \\ A'' & B'' & C' & k'A'' \end{vmatrix}^2 + m \begin{vmatrix} x & y & z & 0 \\ A & B & C & B \\ A' & B' & C' & kB' \\ A'' & B'' & C'' & k'A'' \end{vmatrix}^2 + \text{etc.}$$

$$= lx^2 + my^2 + nz^2 + 2pxy + 2qxz + 2ryz \equiv f, \qquad (47)$$

which shows that the function f is invariant in form under the normal form of T_1 (and also of T) and has the same value at a pair of corresponding points of the plane. It is therefore invariant in value at an invariant point of the plane. The function f vanishes at every point of a certain conic K.

We must now examine the relation of the conic K to the invariant triangle of T. Evidently the position of the conic is not independent of the position of the invariant points A, A', A''. Since the properties of pole and polar with respect to a conic are projective properties, it follows that if a point P and a conic K are invariant under T_1 the polar of P is also invariant. Suppose the invariant point A is not on the conic K; its polar p is an invariant line of T and cuts k in two points, B and C, which are also invariant points. But T leaves invariant only these points; hence B and C coincide with A' and A'' respectively. The polar of A' (or A'') which is on the conic is the tangent at A' (or A''), and this passes through A. Hence K passes through two vertices of the invariant triangle and at these points touches two sides of the invariant triangle (see Fig. 25). Evidently the conic K may have any one of three positions with reference to the invariant triangle. Each side in turn may be the chord of contact with the other two sides as tangents.

236. *The k-Relation for $G_3(K)$.* We found in Art. 132 that the determinant Δ of the normal form of T is $kk'\Delta'^3$; we also found in Art. 210 that the determinant of T in $G_3(K)$ is $\Delta = 1$. Hence we have for every collineation in $G_3(K)$ the condition $kk'\Delta' = 1$. When A is the invariant point of T not on the conic, we have $\Delta' = 1$; substituting this value in $kk'\Delta' = 1$, we get

$$kk' = 1. \tag{48}$$

This shows that the cross-ratios along the two invariant tangent lines to k from the invariant point A have reciprocal values. In the same way when A' (or A'') is not on the conic, we get

$$\frac{k'}{k}\cdot\frac{1}{k} = 1 \ \left(\text{or } \frac{k}{k'}\cdot\frac{1}{k'} = 1\right). \tag{48'}$$

These relations have the same interpretation along invariant tangent lines to the conic. Hence in $G_3(K)$ the cross-ratios k and k' are not independent of each other.

Equation X of Art. 179½ reduces to the identity of $1 = 1$ for the group $G_3(K)$ and we can get from it no relation between the k's of T, T_1 and T_2. Hence for the group $G_3(K)$ the second k-relation turns out to be not a relation between the k's of two components and their resultant, but between the k's of each collineation in the group.

237. *k-Relations for $G_2(AlK)$ and $G_1(A'A''K)$.* The group $G_2(AlK)$ is defined by sets of both linear and quadratic relations, hence the results of the last article and of theorem 27 both hold. Since this group has an invariant lineal element, we have $k_2 = kk_1$ and $k_2' = k'k_1'$; but since $k' = \frac{1}{k'}$, $k_1' = \frac{1}{k_1}$ and $k_2' = \frac{1}{k_2}$, we have only one k-relation, viz.: $k_2 = kk_1$. Similar reasoning applies to the group $G_1(AA'A''K)$ and we reach the same result, viz.: one k-relation $k_2 = kk_1$.

238. *The Insufficiency of the k-Relations.* We have seen that the existence of the k-relations in the subgroup of G_8 is a necessary consequence of the vanishing of the function $\Delta(1)$. Hence we conclude that the existence of a k-relation is a necessary condition for a subgroup of G_8; but we can not infer that it is also a sufficient condition. For if we assume the existence of the k-relation $k_2k_2' = kk'k_1k_1'$ and combine it with equation X, we get $\Delta_2' = \Delta'\Delta_1'$ and this alone is insufficient to restrict the group G_8.

239. We sum up the results of our investigation on the k-relations as follows:

THEOREM 29. A k-relation between the k's of two components and their resultant holds in reducible groups; if the subgroup is irreducible the relation is between the k's of each collineation in the group.

240. *Reduced Normal Form of $G_6(A)$.* Since the group $G_6(A)$ is reducible in the general form we may expect its normal form to possess the same property. The geometrical significance of the reduced form of $G_6(A)$ is that the invariant point coincides with one vertex of the triangle of reference. If T and T_1 have one vertex of their invariant triangles in common, it is evident geometrically that T_2 will also leave the same point invariant. To show this analytically we let T and T_1 have the invariant point A in common and let this common point be chosen as one vertex of the triangle of reference. For example, let the co-ordinates of A be $(0, 0, C)$; then in the normal form of T we must put $A = B = 0$ and in T_1, $A_1 = B_1 = 0$. Substituting these values in the equations I-IX, Art. 179, we see that the right-hand sides of III and VI vanish and the right-hand side of IX reduces to

$$C \begin{matrix} A' & B' \\ A'' & B'' \end{matrix}, \quad C_1 \begin{matrix} A_1' & B_1' \\ A_1'' & B_1'' \end{matrix}.$$

We now have the three following equations from which to determine the values of A_2, B_2 and C_2:

$$\begin{vmatrix} A_2 & B_2 & A_2 \\ A_2' & B_2' & k_2A_2' \\ A_2'' & B_2'' & k_2'A_2'' \end{vmatrix} = 0, \quad \begin{vmatrix} A_2 & B_2 & B_2 \\ A_2' & B_2' & k_2B_2' \\ A_2'' & B_2'' & k_2'B_2'' \end{vmatrix} = 0,$$

$$\begin{vmatrix} A_2 & B_2 & C_2 \\ A_2' & B_2' & k_2C_2' \\ A_2'' & B_2'' & k_2'C_2'' \end{vmatrix} = CC_1 \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \cdot \begin{vmatrix} A_1' & B_1' \\ A_1'' & B_1'' \end{vmatrix} \tag{49}$$

Expanding these three determinants along the top row and collecting coefficients we get

$$\begin{aligned} m_1 A_2 + n_1 B_2 &= 0, \\ m_2 A_2 - n_2 B_2 &= 0, \\ m_3 A_2 + n_3 B_2 + C_2 C &= p ; \end{aligned}$$

where m_1, etc., are all second order determinants in A_2', B_2', etc. Solving the above equations we have $A_2 = 0$ and $B_2 = 0$ unless $\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix} = 0$. But the value of $\begin{vmatrix} m_1 & n_1 \\ m_2 & n_2 \end{vmatrix}$ is readily found to be $(1 - k_2)(1 - k_2') \begin{vmatrix} A_2' & B_2' \\ A_2'' & B_2'' \end{vmatrix}^2$, which does not vanish so long as T_2 is of type I. Hence we have $A_2 = 0$ and $B_2 = 0$. Therefore T_2 also leaves invariant the point $(0, 0, C)$ and its normal form reduces also to the same reduced form as T and T_1.

Equations IX and X reduce respectively to

$$C_2 \begin{vmatrix} A_2' & B_2' \\ A_2'' & B_2'' \end{vmatrix} = C \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix} \cdot C_1 \begin{vmatrix} A_1' & B_1' \\ A_1'' & B_1'' \end{vmatrix}, \tag{IX$'$}$$

and

$$k_2k_2'C''^3_2 \begin{vmatrix} A_2' & B_2' \\ A_2'' & B_2'' \end{vmatrix}^3 = kk'C^3 \begin{vmatrix} A' & B' \\ A'' & B'' \end{vmatrix}^3 \cdot k_1k_1'C_1^3 \begin{vmatrix} A_1' & B_1' \\ A_1'' & B_1'' \end{vmatrix}^3 . \tag{X$'$}$$

Dividing X′ by the cube of IX′ we get

$$k_2k_2' = kk'k_1k_1', \tag{43}$$

which is the same k-relation that we found in Art. 231.

241. *Reduced Normal Form of* $G_6(l)$. If T and T_1 have a common invariant line, then T_2 will have the same invariant line. Let the line l'', joining A' and A'', be the common invariant line of T and T_1 and let this line be taken as the line $z = 0$ of the triangle of reference. Thus we have $C' = C'' = 0$ and $C_1' = C_1'' = 0$ in the normal forms of T and T_1 respectively. Substituting these values in equations I, IX, we find that the right-hand sides of equations VII and VIII vanish and that IX reduces to the same form as in Art. 240. Solving as before equations VII, VIII, and IX, for C_2', C_2'', we find $C_2' = C_2'' = 0$. Equations IX and X reduce to IX′ and X′ respectively of Art. 240, and hence we have the same k-relations as before, viz.: $k_2k_2' = kk'k_1k_1'$.

242. *Reduced Normal Form of Other Subgroups of* G_8. Let T and T_1 have in common the invariant point A and the invariant line l; the group is then $G_5(Al)$, and T_2 will also leave A and l invariant. Making $A=B=B'=0$ and $A_1=B_1=B_1'=0$ in equations I-IX, we find $A_2=B_2=B_2'=0$. Equations V, IX, and X reduce respectively to

$$k_2' A_2' B_2'' C_2 = k' A' B'' C \;\; k_1' A_1' B_1'' C_1. \qquad \text{V}'.$$

$$A_2' B_2'' C_2 = A' B'' C \;\; A_1' B_1'' C_1. \qquad \text{IX}'.$$

$$k_2 k_2' A_2' B_2'' C_2 = kk' A' B'' C \;\; k_1 k_1' A_1' B_1'' C_1. \qquad \text{X}'.$$

Dividing V′ by IX′ we get $k_2' = k' k_1'$; dividing X′ by V′ we get $k_2 = k k_1$.

In like manner we can find the reduced normal forms of each of the remaining groups in the list of Art. 209 and 216, and verify in this way their k-relations as given in Art. 234, etc.

We have thus verified, by means of the reduced normal forms of the reducible subgroups of G_8, the theory of the k-relations stated in Theorems 27 and 28. The group $G_3(K)$ being irreducible the theory of its k-relations can not be verified in this way; it will, however, be amply confirmed later.

§ 7. Fundamental Groups, One-Parameter Groups and their Path-Curves.

243. We shall now consider for each type of collineation a certain group which we shall call the fundamental group of that type. We have shown, Chapter II, Theorem 18, that each type of collineation has its own characteristic invariant figure. We have also shown that all collineations of a given type which leave invariant this characteristic figure form a group, the so-called fundamental group for that type. The fundamental group of type I is the group $G_2(AA'A'')$, Art. 197, whose invariant figure is the triangle $(AA'A'')$. The fundamental group of type II is $G_2'(AA'l)$, Art. 226; that of

type III is $G_3''(Al)$, Art. 227; that of type IV is $H_1(A_1 l)$, Art. 228; that of type V is $H_1'(Al)$, Art. 229.

The fundamental groups of types I, II, and III each have more than one parameter and break up into one-parameter subgroups. The fundamental group of types IV and V are one-parameter groups. Each one-parameter group of collineations leaves invariant a family of curves called the Path-Curves of the group. The property of these path-curves will also be investigated in the present section.

The efficient instrument for the investigation of these fundamental groups, and in fact of any subgroup of G_8, is the normal form of the collineation T. We shall, therefore, in this and in §§ 8 and 9, make constant use of the normal forms of the various types of collineations.

A. FUNDAMENTAL GROUPS OF TYPES IV AND V.

244. *Fundamental Group of Type IV.* A perspective collineation S of type IV leaves invariant a point A (the vertex), and a line l (the axis) not through A, all lines of the plane through A and all points on the line l, Fig. 14, IV. It is further characterized by a constant, k, the cross-ratio of the one-dimensional transformations along each of the invariant lines through A. The equations of S may be reduced to the canonical form, Art. 151,

$$S: \begin{array}{l} \rho x_1 = kx \\ \rho y_1 = ky \\ \rho z_1 = z. \end{array} \tag{50}$$

Since k may have any value whatever it follows that there is a set of ∞^1 perspective collineations leaving the same figure invariant. Let S_1 be a second collineation of the same set. The equations of S_1 may be written

$$S_1: \begin{array}{l} \rho_1 x_2 = k_1 x_1 \\ \rho_1 y_2 = k_1 y_1 \\ \rho_1 z_2 = z_1. \end{array} \tag{50'}$$

Eliminating $(x_1 y_1 z_1)$ from the equations of S and S_1 we get the equations of their resultant S_2 in the form

$$S_2: \begin{array}{l} \rho_2 x_2 = k_2 x_1 \\ \rho_2 y_2 = k_2 y_1 \\ \rho_2 z_2 = z, \end{array} \text{ where } k_2 = kk_1. \tag{50''}$$

Hence S_2 belongs to the same set as S and S_1 and the set has the first group property.

The inverse of S is found by solving the equations of S for x and y. Thus

$$S^{-1}: \begin{aligned} \rho_3 x &= k^{-1} x_1 , \\ \rho_3 y &= k^{-1} y_1 , \\ \rho_3 z &= z_1 . \end{aligned}$$

Hence the inverse of S is also in the set and the set possesses the second group property. The set therefore forms a group designated by $H_1(A, l)$; it is a one-parameter group, the cross-ratio k being the parameter of the group.

THEOREM 30. The fundamental group of type IV consists of all collineations of type IV having the same vertex and axis: it is a one-parameter group whose parameter is the cross-ratio k.

245. *Properties of the Group* $H_1(A, l)$. This group $H_1(A, l)$ has essentially the same properties as the one-parameter group $G_1(A'A)$ of our one-dimensional transformations of points on a line (Chap. I, Art. 27). It is needless to repeat the statement of those properties. In both groups the law of combination of parameters is $k_2 = kk_1$ and the parameters vary in precisely the same way. The transformations of the two groups $G_1(A'A)$ and $H_1(A, l)$ have a one-to-one correspondence and because of this property are said to be *holoedrically isomorphic.*

246. *Fundamental Group of Type V.* A perspective collineation S' of type V leaves invariant a point A (the vertex), a line l (the axis) through A, every line of the plane through A and every point on the line l, Fig. 14, V. Along every invariant line through A there is a one-dimensional parabolic transformation having only one invariant point, viz., A.

The canonical form of S' may be written (Art. 152):

$$S' : \qquad x_1 = \frac{x}{1+tx}, \quad y_1 = \frac{y}{1+tx} ; \tag{51}$$

where the vertex A is the origin and the axis l is the x-axis. Since t may have any value whatever, we see that there is a set of ∞^1 elations all having the same invariant figure. Let another elation of the same set be S_1' having the constant t_1. The equations of S_1', are

$$S_1' : \quad x_2 = \frac{x_1}{1+t_1x_1}, \quad y_2 = \frac{y_1}{1+t_1x_1}. \tag{51'}$$

Eliminating x_1 and y_1 we get the equations of the resultant S_2' as follows:

$$S_2' : \quad x_2 = \frac{x}{1+t_2x}, \quad y_2 = \frac{y}{1+t_2x}; \tag{51''}$$

where $t_2 = t + t_1$. Thus S_2' belongs to the same set as S' and S_1'. The inverse of S' is found by solving the equations of S' for x and y. Thus

$$S'^{-1} : \quad x = \frac{x_1}{1-tx_1}, \quad y = \frac{y_1}{1-tx_1}. \tag{51'''}$$

The inverse of S' is also in the set; both group properties are satisfied and hence the set is a group designated by $H_1'(Al)$. It is a one-parameter group, t being the parameter.

THEOREM 31. **The fundamental group of elations in the plane is a one-parameter group, whose parameter is t.**

247. *Properties of the Group $H_1'(Al)$.* The group $H_1'(Al)$ of elations in the plane and the group $G_1'(A)$ of one-dimensional parabolic transformations, Art. 29, are holoedrically isomorphic. They each contain a parameter t, which combines according to the parabolic law, $t + t_1 = t_2$. Hence the properties of $H_1'(Al)$ need not be discussed in detail but may be inferred at once from the properties of $G_1'(A)$, stated in Chap. I, Art. 29.

B. FUNDAMENTAL GROUP OF TYPE I AND ITS SUBGROUPS.

248. *Fundamental Group of Type I.* It was shown in equation (21) Chap. II, Art. 148, that, when the invariant triangle of a collineation of type I is taken as the triangle of reference, the equation of a collineation T reduces to the form:

$$T : \quad \begin{aligned} \varsigma x_1 &= kx, \\ \varsigma y_1 &= k'y, \\ \varsigma z_1 &= z. \end{aligned} \tag{52}$$

where k and k' are the cross-ratios of the one-dimensional transformation along the sides AA' and AA'' respectively. These equations contain two independent parameters k and k'. Giving to k and k' all possible values, we get a double infinity of collineations all leaving the same triangle, $AA'A''$, invariant. We wish to show that these ∞^2 collineations form a group.

Let us take any two collineations T and T_1 from this set and form their resultant T_2. It is evident, geometrically, that T_2 belongs to the same set; for if T and T_1 each leave the triangle $AA'A''$ invariant, their resultant T_2 must also leave it invariant and hence must belong to the same set. This may also be shown analytically. Let the equations of T and T_1 be respectively:

$$T: \begin{array}{l} \rho x_1 = kx, \\ \rho y_1 = k'y, \\ \rho z_1 = z, \end{array} \quad \text{and } T_1: \begin{array}{l} \rho_1 x_2 = k_1 x_1, \\ \rho_1 y_2 = k'_1 y_1, \\ \rho_1 z_2 = z_1. \end{array} \tag{52'}$$

Eliminating x_1, y_1, z_1 we get T_2. Thus

$$T_2: \begin{array}{l} \rho_2 x_2 = k_2 x, \\ \rho_2 y_2 = k'_2 y, \\ \rho_2 z_2 = z, \end{array} \quad \text{where } k_2 = kk_1 \text{ and } k'_2 = k' k'_1. \tag{52''}$$

Since T_2 is of the same form as T and T_1, it belongs to the same set and the first group property is established.

The inverse of T is

$$T^{-1}: \begin{array}{l} \rho^{-1} x = k^{-1} x_1, \\ \rho^{-1} y = k^{-1} y_1, \\ \rho^{-1} z = z_1. \end{array} \tag{52'''}$$

This collineation is also in the set and hence the second group property is established. Thus the set forms a two-parameter group, which will be designated by $G_2(AA'A'')$.

THEOREM 32. The fundamental group of type I consists of all collineations of type I having the same invariant triangle; it is a two-parameter group whose parameters are the cross-ratios k and k'.

249. *One-parameter Subgroups of* $G_2(AA'A'')$. We shall now show that the group $G_2(AA'A'')$ contains ∞^1 one-parameter subgroups. Let k' be replaced by k^r. The collineation

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T is now characterized by two constants, k and r, and the group $G_2(AA'A'')$ has the two parameters, k and r.

Let r be fixed and let k alone vary. In this way we select from the group $G_2(AA'A'')$ a set of ∞^1 collineations. We wish to show that this set forms a one-parameter group.

To show this take two collineations T and T_1 from the set characterized respectively by the constants (k, r) and (k_1, r) and find their resultant. From equations (52″) we have $k_2 = k k_1$ and $k_2' = k' k_1'$; setting $k' = k^r$ and $k_1' = k_1^r$ in these equations we get $k_2 = kk_1$ and $k_2' = k^r k_1^r = (kk_1)^r = k_2^r$. Thus the resultant T_2 is characterized by the constants (k_2, r). Hence the resultant belongs also to the set and the first group property is established for the set. The inverse of $T(k, r)$ is $T(k^{-1}, r)$; this also belongs to the set and the set is therefore made up of pairs of inverse collineations. Both group properties are therefore established and the set is a one-parameter group. This one-parameter group $G_1(AA'A'')_r$ is a subgroup of $G_2(AA'A'')$. The group $G_2(AA'A'')$ contains ∞^1 such subgroups, one for each value of r.

THEOREM 33. The fundamental group of type I, $G_2(AA'A'')$, contains ∞^1 one-parameter subgroups $G_1(AA'A'')_r$; each subgroup is characterized by a constant r, and its variable parameter is k.

250. *Properties of* $G_1(AA'A'')_r$. This one-parameter group has properties very similar to those of the group $G_1(AA')$ of one-dimensional transformations discussed in Chap. I, Art. 27. The identical collineation is in the group and is given by $k = 1$, the infinitesimal collineation of the group is given by $k = 1 + \delta$, Art. 27, where δ is infinitesimally near to zero. Corresponding to $k = 0$ and $k = \infty$ there are two pseudo-collineations. The group is, properly speaking, discontinuous for these values of parameter k. It is continuous for all values of k except $k = 0$ and $k = \infty$.

251. *Path-curves of* $G_1(AA'A'')_r$. We wish to investigate the effect on a point P of the plane of all the collineations of

the group $G_1(A'AA'')_r$. The infinitesimal collineation of the group

$$\rho x_1 = x + \delta x,$$
$$\rho y_1 = y + r\delta y,$$
$$\rho z_1 = z,$$

moves P to P_1; applied again it moves P_1 to P_2; applied an infinite number of times it moves P along a certain curve in the plane called a *path-curve.* There are ∞^1 of these path-curves so situated that every point in the plane lies on one and only one of them. The equation of this family of path-curves is found by eliminating k from the equations of the group, viz.:

$$\frac{x_1}{z_1} = k\frac{x}{z}; \quad \frac{y_1}{z_1} = k^r\frac{y}{z}. \tag{52}$$

Eliminating k we get

$$x^r y^{-1} z^{1-r} = x_1^r y_1^{-1} z_1^{1-r} = C, \text{ or } x^r z^{1-r} = Cy. \tag{53}$$

Thus the path-curves of the group $G_1(AA'A'')r$ are curves of order r and form a family whose parameter is C. Fig. 24.

THEOREM 34. When the invariant triangle is taken as the triangle of reference, the family of path-curves of a one-parameter group $G_1(AA'A'')_r$ is given by $x^r z^{1-r} = Cy$, where r is a constant.

252. *Geometric Meaning of r.* It is not difficult to determine, from the equation of the family of path-curves, the geometric meaning of r. Take any point $P=(x_1, y_1, z_1)$ and draw a tangent at P to the path-curve through P and join P to the vertices of the triangle $(AA'A'')$. The equation of the tangent to the path-curve at P is

$$PT \ : \ x(Cr)y_1z_1 - yx_1z_1 + z(1-Cr)x_1y_1 = 0.$$

The lines PA, PA', and PA'' are given by the equations

$$PA \ : \ yx_1 - xy_1 = 0,$$
$$PA' : \ yz_1 - zy_1 = 0,$$
$$PA'' : \ yz_1 - zx_1 = 0.$$

For $C=1$, the cross-ratio of the pencil $P(TAA'A'')$ is found to be r. It is evident that this cross-ratio r is a constant for every point on every path-curve of the group $G_1(AA'A'')_r$.

THEOREM 35. The constant r in the one-parameter group $G_1(AA'A'')_r$ is the cross-ratio of the pencil of four lines formed by a tangent to one of the path-curves and the three lines drawn from its point of contact to the vertices of the invariant triangle.

253. *Path-curves are Straight Lines when* $r=1, 0, \infty$. The path-curves of a one-parameter subgroup of $G_2(AA'A'')$ reduce to straight lines for three values of r, viz.: $r=(1, 0, \infty)$. Let $r=1$ in $x^r z^{1-r}=Cy$, and we get a pencil of lines $x=Cy$ through A; let $r=0$ and we get $z=Cy$, a pencil of lines through A'; let $r=\infty$ and we get $x=Cz$, a pencil of lines through A''.

When $r=1$ the cross-ratio of the one-dimensional transformations along the line AA' and AA'' are equal and hence that along the line $A'A''$ is unity, thus this last transformation is identical; hence every point on $A'A''$ is an invariant point. Consequently every line through A is an invariant line; hence the subgroup of $G_2(AA'A'')$ for $r=1$ is a group of perspective collineations, having A for the vertex and $A'A''$ for the axis. In like manner we see that when $r=0$ and ∞, we have subgroups of perspective collineations and vertices A' and A'', and axes AA'' and AA' respectively.

THEOREM 36. The group $G_2(AA'A'')$ contains three subgroups of perspective collineations, viz., the subgroups for which $r=1, 0, \infty$. For these three subgroups the path-curves are straight lines.

254. *Path-curves are Conics when* $r=-1, 2, 1/2$. There are three other specially important subgroups of $G_2(AA'A'')$; these correspond to the values $r=-1, 2, 1/2$. Putting $r=2$ in $x^r z^{1-r}=Cy$, we get $yz=Cx^2$; the path-curves reduce in this case to a system of conics having double contact at A and A''. $A'A$ and $A'A$ are common tangents to the pencil of conics and $AA'A''$ is the chord of contact. In like manner, when $r=-1$, the path-curves are $xy=Cz^2$; when $r=1/2$, the path-curves are $xz=Cy^2$. These are also pencils of conics having double contact; in each case two sides

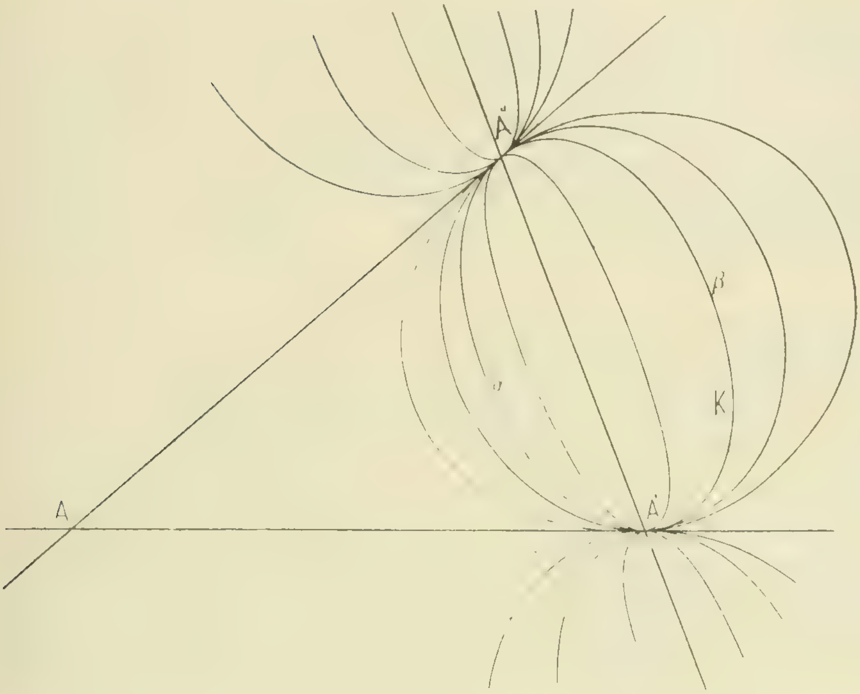

FIG. 24.

of the invariant triangle are common tangents and the third side is the chord of contact.

THEOREM 37. The group $G_2(AA'A'')$ contains three special subgroups for which the path-curves are conics having double contact; these are the subgroups for which $r = -1, 2, 1/2$.

255. *Rational and Irrational Values of* r. The subgroups of $G_2(AA'A'')$ for which r is rational have important properties not possessed by the subgroups for which r is irrational. The equation of the family of path-curves is

$$x^r z^{1-r} = Cy. \qquad (53)$$

These curves are algebraic or transcendental according as r is a rational or irrational number. The vertices of the invariant triangle are singular points on all curves of the family when r is irrational; one vertex is a singular point when r

is rational, except when the path-curves are straight lines or conics.

For any integral value of r, except the special values $1, 0, \infty, -1, 2$, there are six subgroups for which the path-curves are algebraic curves of the order r, viz.: $r, 1/r, 1-r, 1/(1-r), r/(r-1), (r-1)/r$. Thus for $r=3$ the path-curves are cubics, viz.:

$$x^3 = Cyz^2;$$

the path-curves are also cubics for $r = 1/3, -2, -1/2, 2/3, 3/2$, and for no other values of r.

256. *Involutoric Collineations in* $G_2(AA'A'')$. The group $G_2(AA'A'')$ contains three subgroups of perspective collineations, viz.: when $r=1, 0, \infty$. Each of these groups of perspective collineations contains an involutoric collineation and hence $G_2(AA'A'')$ contains three distinct involutoric collineations. The values of k and k' which correspond to these involutoric collineations are as follows: When $k=-1$ and $k'=-1$, the involutoric collineation has its vertex at A and $A'A''$ for axis; for the pair of values $(-1, 1)$ the collineation is involutoric, having A' for vertex and AA'' for axis; for the pair of values $(1, -1)$ the collineation is again involutoric having A'' for vertex and AA' for axis.

The equation $k^1 = \pm k^r$ is satisfied by the pair of values $k=-1$, $k^r = 1$ when r is rational with even numerator and odd denominator; it is satisfied by the pair of values $(-1, -1)$ when r is rational with odd numerator and odd denominator; it is satisfied by the pair of values $(1, -1)$ when r is rational with odd numerator and even denominator. The equation is not satisfied by either of these pairs of values when r is an irrational number. Hence the involutoric collineation given by the pair of values $(-1, 1)$ belongs to every subgroup of $G_2(AA'A'')$ for which r is rational with even numerator and odd denominator; the involutoric collineation given by the pairs of values $(-1, -1)$ and $(1, -1)$ are contained respectively in every subgroup for which r is rational with odd numerator and odd denominator and odd numerator and even denominator respectively. None of these three involutoric collineations can belong to a group $G_1(AA'A'')_r$ for which r is irrational.

THEOREM 38. Every one-parameter group $G_1(ABC)_r$ for which r is rational contains an involutoric perspective collineation; no such group for which r is irrational contains an involutoric perspective collineation.

C. FUNDAMENTAL GROUP OF TYPE II AND ITS SUBGROUP.

257. *Fundamental Group of Type II.* The canonical form of a collineation T' of type II has to be, Art. 149:

$$T' \ : \quad x_1 = \frac{kx}{1+\frac{k-1}{A}x+ty}, \quad y_1 = \frac{y}{1+\frac{k-1}{A}x+ty}; \qquad (54)$$

when the invariant lines of the figure are the lines $x=0$ and $y=0$, and the invariant points are the origin and the point $(A,0)$. Equations (54) may also be written:

$$\frac{x_1}{y_1} = k\frac{x}{y} \text{ and } \frac{1}{y_1} = \frac{1}{y} + \frac{k-1}{A}\frac{x}{y} + t.$$

A second collineation T_1' with the same invariant figure may be written:

$$T_1' \ : \quad \frac{x_2}{y_2} = k_1\frac{x_1}{y_1} \text{ and } \frac{1}{y_2} = \frac{1}{y_1} + \frac{k_1-1}{A}\frac{x_1}{y_1} + t_1. \qquad (54')$$

Eliminating x_1 and y_1 from the equations of T' and T_1' we get the resultant T_2 in the form

$$T_2' \ : \quad \frac{x_2}{y_2} = k_2\frac{x}{y} \text{ and } \frac{1}{y_2} = \frac{1}{y} + \frac{k_2-1}{A}\frac{x}{y} + t_2, \qquad (54'')$$

where $k_2 = kk_1$ and $t_2 = t + t_1$. The inverse of T' is found by solving equations (14) for x and y;

$$T'^{-1} : \quad \frac{x}{y} = k^{-1}\frac{x_1}{y_1} \text{ and } \frac{1}{y} = \frac{1}{y_1} + \frac{k^{-1}-1}{A}\frac{x_1}{y_1} - t. \qquad (54''')$$

Hence both group properties are established for the set of collineations of type II leaving the figure $(AA'l)$ invariant. The set is therefore a two-parameter group designated by $G_2'(AA'l)$, the parameters being k and t.

THEOREM 39. The fundamental group of type II consists of all collineations of type II which leave invariant the same figure $AA'l$; it is a two-parameter group, the parameters being k and t.

258. *One-parameter Subgroups of* $G_2'(AA'l)$. We proceed to show that the group $G_2'(AA'l)$ breaks up into ∞^1 one-parameter subgroups. The two independent parameters of $G_2'(AA'l)$ are k and t and the laws of combinations of these parameters are expressed by $k_2 = kk_1$ and $t_2 = t + t_1$. Let us set $k = a^t$, where a is a constant, and let t be the independent variable. In this way we select from the group $G_2'(AA'l)$ a set of ∞^1 collineations. Let T' and T_1' be two collineations of this set, characterized respectively by the parameters (a^t, t) and (a^{t_1}, t_1). Their resultant T_2' has the parameters (a^{t_2}, t_2), where $t_2 = t + t_1$. The inverse of t has the parameters $(a^{-t}, -t)$. Both group properties are satisfied by the collineations of the set satisfying the relations $k = a^t$, and the set is a one-parameter group designated by $G_1'(AA'l)_a$. Within the group $G_2'(AA'l)$ there are ∞^1 such subgroups, one for each value of a (except $a = 0$).

THEOREM 40. The fundamental group $G_2'(AA'l)$ breaks up into ∞^1 one-parameter subgroups $G_1'(AA'l)_a$.

259. *Path-curves of* $G_1'(AA'l)_a$. The effect upon a point P of the plane of all the collineations of the group $G_1'(AA'l)_a$ is to move it along its path-curve. The equation of the family of path-curves of the group $G_1'(AA'l)_a$ is found by eliminating t from the pair of equations

$$\frac{x_1}{y} = a^t \frac{x}{y} \text{ and } \frac{1}{y_1} = \frac{1}{y} + \frac{a^t - 1}{A}\frac{x}{y} + t.$$

Eliminating we get

$$\log_a \frac{x_1}{y_1} - \left(\frac{1}{y_1} - \frac{x_1}{Ay_1}\right) = \log_a \frac{x}{y} - \left(\frac{1}{y} - \frac{x}{Ay}\right) = C.$$

whence $x = Cya^{\frac{A-x}{Ay}}$. (56)

The curves of this family are transcendental curves; they all pass through the points A and A' and have these points as singular points.

THEOREM 41. The family of path-curves of the group $G_1'(AA'l)_a$ are transcendental curves and are given by the equation $x = Cya^{\frac{A-x}{Ay}}$.

260. *Two Special Subgroups of* $G_2'(AA'l)$. Among the ∞^1 subgroups of $G_2'(AA'l)$, two are of special importance and require attention. When $a = 1$, we have $k = 1^t = 1$. This signifies that the one-dimensional transformations along AA' and through A are both identical transformations, and hence all points on l and all lines through A are invariant under all collineations of the group. The one-dimensional transformation in the pencil through A' is parabolic and hence along all the invariant lines through A the one-dimensional transformations are also parabolic. This particular subgroup of $G_2'(AA'l)$ is therefore a group of elations $H_1'(Al')$ (l' being the line AA'). When $t = 0$ and k alone varies (this is equivalent to $a = \infty$), the one-dimensional transformations along l and through A' are both identical. In this case all points on l and all lines through A' are invariant under all collineations of the group. The group $G_1'(AA'l)_{a=\infty}$ is, therefore, the group of perspective collineations of type IV $H_1(A'l)$.

The equations of the family of path-curves $x = Cya^{\frac{A-x}{Ay}}$ reduces to $x = Cy$ for $a = 1$ and to $y = C'(A - x)$ for $a = \infty$.

THEOREM 42. The two-parameter group $G_2'(AA'l)$ contains one subgroup of collineations of type V, viz.: $H_1'(Al')$ for $a = 1$; and one subgroup of collineations of type IV, viz.: $H_1(A'l)$ when $a = \infty$.

261. *Properties of the Group* $G_1'(AA'l)_a$. The parameter of the group $G_1'(AA'l)_a$ is t and the law of combination of parameters is $t_2 = t + t_1$. Consequently the properties of the group are quite similar to those of the parabolic group of one-dimensional transformations $G_1'(A)$ (Chap. 1, Art. 29).

Two groups $G_1'(AA'l)_a$ and $G_1'(AA'l)_a$ have in common the identical and the pseudo-collineations corresponding to $t = 0$ and $t = \infty$. We wish to ascertain if they have any other collineation in common. If a collineation whose constants are k and t belongs to both the above groups, then k and t must satisfy both equations $k = a^t$ and $k = a_1^t$. This is possible only when $a = a_1$. Hence the two groups have no collineation in common, save the identical and the pseudo-collineations.

D. FUNDAMENTAL GROUP OF TYPE III AND ITS SUBGROUPS.

262. *Fundamental Group of Type III.* The canonical form of a collineation T'' of type III has been found, Art. 150, to be

$$T'' : \begin{cases} \dfrac{x_1}{y_1} = \dfrac{x}{y} + 2at, \\[2ex] \dfrac{1}{y_1} = \dfrac{1}{y} + t\dfrac{x}{y} + (at^2 + ht) \end{cases} ; \qquad (57')$$

the origin is the only invariant point and the x-axis the only invariant line. T'' depends upon three constants, a, h, t, and hence there is a set of ∞^3 collineations of type III having the same fundamental invariant figure. Let T_1'' be a second collineation of the same set given by the equations

$$T_1'' : \begin{cases} \dfrac{x_2}{y_2} = \dfrac{x_1}{y_1} + 2a_1t_1, \\[2ex] \dfrac{1}{y_2} = \dfrac{1}{y_1} + t_1\dfrac{x_1}{y_1} + a_1t_1^2 + h_1t_1 \end{cases} . \qquad (57')$$

The resultant of T'' and T_1'' is found by the elimination of x_1 and y_1 to be

$$T_2'' : \begin{cases} \dfrac{x_2}{y_2} = \dfrac{x}{y} + 2a_2t_2 \\[2ex] \dfrac{1}{y_2} = \dfrac{1}{y} + t_2\dfrac{x}{y} + a_2t_2^2 + h_2t_2, \end{cases} \qquad (57'')$$

where

$$\begin{aligned} &t_2 = t + t_1, \\ &a_2t_2 = at + a_1t_1, \\ &a_2t_2 + h_2t_2 = at + 2att_1 + a_1t_1 + ht + h_1t_1 . \end{aligned} \qquad (58)$$

The resultant, T_2'', belongs to the same set as T'' and T_1'', and hence the first group property is established for the set.

The inverse of T'' is gotten by solving the equations of T'' for x and y. Thus:

$$T''^{-1}\ :\quad \begin{aligned} \frac{x}{y} &= \frac{x_1}{y_1} + 2a\,(-t)\,, \\ \frac{1}{y} &= \frac{1}{y_1} - t\,\frac{x}{y} + at^2 - ht\,. \end{aligned} \tag{57'''}$$

The equations of T'' and its inverse differ only in the sign of the parameter t. Hence both group properties are established and the set is a group of three parameters, $G_3''(Al)$.

263. *The Two-parameter Group $G_2''(AlN)$.* Let $T''(aht)$ and $T_1''(ah_1t_1)$ be two collineations of the group $G_3''(Al)$, *i. e.*, let T'' and T_1'' be so chosen that they are characterized by the same constant, a. Then their resultant T_2'' is also characterized by the same constant, a; for if $a_1 = a$ in the three equations (18), then also $a_2 = a$ and they reduce to two, as follows:

$$\begin{aligned} t_2 &= t + t\,, \\ h_2 t_2 &= h\,t + h_1 t_1\,. \end{aligned} \tag{59}$$

Hence in $G_3''(Al)$ if a be kept fixed and h and t be allowed to vary, we select from $G_3''(Al)$ ∞^2 collineations which form a two-parameter group, $G_2''(AlN)$.*

On the other hand, if h or t be kept constant in equations (18), these three equations do not reduce to a smaller number and we have no corresponding subgroups.

THEOREM 43. The fundamental group $G_3''(Al)$ contains ∞^1 two-parameter subgroups $G_2(AlN)$, one for each value of the constant a.

264. *One-parameter Groups $G_1''(AlS)$.** Let $a_1 = a$ and $h_1 = h$ in the system of equations (58); these equations then reduce to a single equation $t_2 = t + t_1$. Hence if a and h are both kept constant and t alone be made to vary, we thus select out of $G_3''(Al)$ ∞^1 collineations which form a one-para-

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*The significance of the symbol $G_2(AlN)$ will be shown in Art. 266.

meter group, $G_1''(AlS)$. The group $G_3''(Al)$ contains ∞_2 such subgroups, one for each pair of values of a and h; the group $G_2''(AlN)$ contains ∞^1 such subgroups, one for each value of h.

THEOREM 44. Each two-parameter group $G_2''(AlN)$ contains ∞^1 one-parameter subgroups $G_1''(AlS)$, one for each value of h.

265. *Path-curves of $G_1''(AlS)$.* The path-curves of the one-parameter group $G_1''(AlS)$ are found by eliminating the parameter t from the equations of the group:

$$T'' \quad : \quad \begin{aligned} \frac{x_1}{y_1} &= \frac{x}{y} + 2at, \\ \frac{1}{y_1} &= \frac{1}{y} + t\frac{x}{y} + at^2 + ht. \end{aligned} \tag{57}$$

Eliminating t we get

$$\frac{1}{y_1} - \frac{x_1^2}{4ay_1^2} - \frac{hx_1}{2ay_1} = \frac{1}{y} - \frac{x^2}{4ay^2} - \frac{hx}{2ay_1} = C,$$

or

$$x^2 + 2hxy + 4aCy^2 = 4ay. \tag{60}$$

The path-curves of the group $G_1''(AlS)$ are therefore a family of conic sections.

From the equations of the family of conics we see that they all pass through the origin A and have the line l or $y = 0$, for a tangent. Two conics of the family have no point of intersection except the origin; in fact the conics all have contact of the third order at the origin. They therefore form a pencil S of conics through four coincident points. Fig. 25.

THEOREM 45. The path-curves of the group $G_1''(AlS)$ are the conics of a pencil S having contact of the third order at the invariant point A.

266. *Geometric Meaning of the Constant a.* It was shown in Chapter II, Art. 139, that a collineation of type III is the limiting form of a collineation of type I, when the invariant triangle $(AA'A'')$ shrinks to a point. In this case in the equation $k' = k^r$ we have $r = 2$ and the path-curves of the group $G_1(AA'A'')$ are conics having double contact at A and

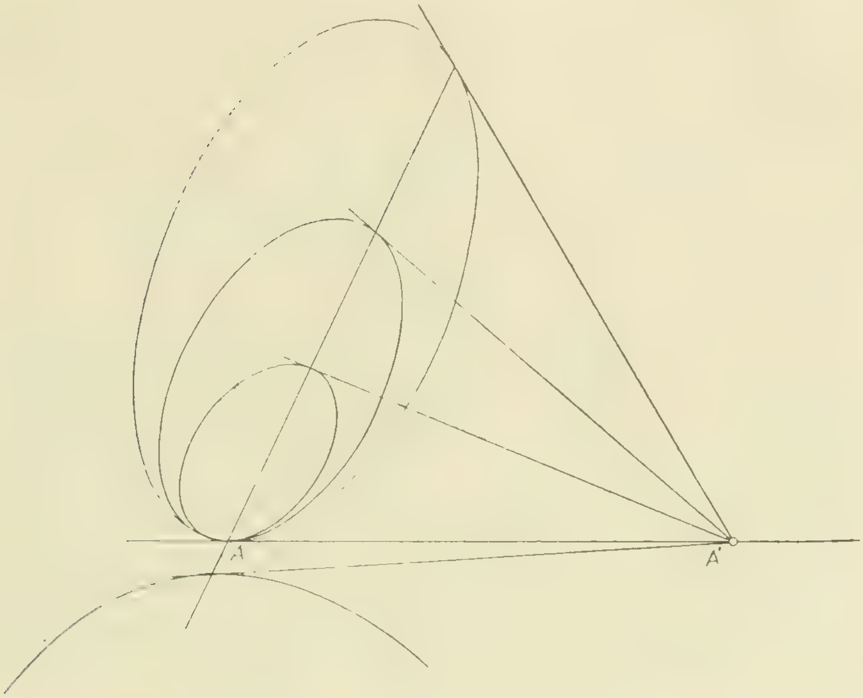

FIG. 25.

A'' and the lines AA' and $A'A''$ as common tangents, Fig. 24. When the collineations of type I in the group $G_1(AA'A'')$ pass over into collineations of type III as their limiting forms as A' approaches A, the pencil of conics having double contact at A and A'' becomes the pencil S of conics having contact of the third order at A.

The constant a designates (Chapter II, Art. 139) the limit of $\frac{l^2}{m}$ as the collineation of type I approaches type III. From the equations

$$l = \left((A'-A)^2 + (B'-B^2) \right)^{1/2},$$

and

$$lm = \begin{vmatrix} A & B & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix} \equiv \begin{vmatrix} A'-A & B'-B \\ A''-2A'+A & B''-2B'+B \end{vmatrix},$$

we have

$$\frac{l^2}{m} = \frac{\left\{ (A'-A)^2 + (B'-B)^2 \right\}^{3/2}}{\begin{vmatrix} A'-A & B'-B \\ A''-2A'+A & B''-2B'+B \end{vmatrix}} \qquad (61)$$

$$= \frac{\left(\triangle x^2 + \triangle y^2\right)^{3/2}}{\triangle x \triangle^2 y - \triangle y \triangle^2 x}.$$

Passing to the limit we have

$$a = lim \frac{l^2}{m} = \frac{\left(dx^2 + dy^2\right)^{3/2}}{dx\, d^2 y - dy\, d^2 x} = R,^*$$

where R is the radius of the limiting circle through three consecutive points. If the three points $AA'A''$ be on a curve, then R is the radius of curvature at the limiting point A. In this case the points $AA'A''$ do not lie in the path-curves of the group $G_1'(AA'A'')_2$, but are the vertices of the triangle formed by two tangents and their chord of contact. Now in the limit the radius of the circle circumscribing the triangle formed by two consecutive tangents and their chord is one-half the radius of curvature.† Hence a is half the radius of curvature at A of each conic of the family of path-curves of the group $G_1''(AlS)$.

This conclusion may be verified directly by calculating the radius of curvature at the origin of the conic

$$x^2 + 2hxy + 4aCy - 4ay = 0.$$

We readily find the radius of curvature at the origin to be $2a$ and thus independent of C, the parameter of the family of conics.

THEOREM 46. The constant a, in the group $G_1''(AlS)$ is half the common radius of curvature of the path-curves of the group at this common point of contact, a.

267. *Geometric Meaning of* h. To find the geometric meaning of h we find the equation of the polars of a point A' on the x-axis with respect to the pencil of conics given by

$$x^2 + 2hxy + 4aCy^2 - 4ay = 0. \qquad (60)$$

The coordinates of A' are $(x', 0)$ and the polar of $(x', 0)$ is

$$x'(x' + hy) = 4ay.$$

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[*] Goursat, *Cours de Math.* vol. I, p. 490.

[†] Salmon, *Conic Sections*, 6th ed., art. 398*a*.

This is independent of C and hence the polar of A' with respect to each conic of the pencil is the same straight line through the origin.

If A' is the point at infinity on $y = 0$, its polar is the common diameter of the pencil of conics. The equation of the common diameter, or line of centers, is found by making $x' = \infty$. We thus get for the line of centers,

$$x + hy = 0 .$$

Hence h is the negative cotangent of the angle which the line of centers makes with the invariant line $y = 0$.

THEOREM 47. The constant h in the group $G_1''(AlS)$ is the negative reciprocal of the slope of the line of centers of the pencil of invariant conics of $G_1''(AlS)$.

268. *Special Subgroups of* $G_3''(Al)$. The fundamental group of type III $G_3''(Al)$ contains two two-parameter subgroups which require special attention, viz.: when $a = 0$ and when $a = \infty$.

First let $a = 0$ in equation (57); we then have

$$T'' : \frac{x_1}{y_1} = \frac{x}{y} \text{ and } \frac{1}{y_1} = \frac{1}{y} + t\frac{x}{y} + ht , \qquad (62)$$

$$\text{or,} \quad x_1 = \frac{x}{1 + tx_1 + hty} \text{ and } y_1 = \frac{y}{1 + tx + hty} .$$

These equations show that all lines through the origin are invariant lines and all points on the line $x + hy = 0$ are invariant points. The collineation T'' reduces in this case to S', a perspective collineation of type II. The system of equations (58) reduce to

$$\begin{aligned} t_2 &= t + t_1 , \\ h_2 t_2 &= ht + h_1 t_1 . \end{aligned} \qquad (59)$$

The group $G_2''(AlN)$ for $a = 0$ is therefore the two-parameter group of elations $H_2'(A)$.

In the second case let $h\,t = t'$; substituting this value of t in the equations of T'' we have

$$T'' : \quad \begin{aligned} \frac{x_1}{y_1} &= \frac{x}{y} + 2\frac{a}{h}t' \\ \frac{1}{y_1} &= \frac{1}{y} + \frac{t'}{h}\frac{x}{y} + \frac{a}{h^2}t'^2 + t'. \end{aligned}$$

Now let both a and h approach ∞ and let the limit of $2\frac{a}{h}$ be A', while t' remains finite. Equations (57) reduce to

$$\begin{aligned} \frac{x_1}{y_1} &= \frac{x}{y} + A't' \\ \frac{1}{y_1} &= \frac{1}{y} + t'. \end{aligned} \tag{63}$$

The collineation represented by these equations leaves invariant every point on the x'-axis and every line through the point $(A'0)$; it is therefore a perspective collineation of type V. Equations (16) represent the two-parameter group of elations $H_2'(l)$.

THEOREM 48. The group $G_3'(Al)$ contains two two-parameter subgroups of type V, viz.: $H_2'(A)$ and $H_2'(l)$; thus $G_2''(AlN)$ for $a = 0 \equiv H_2'(A)$ and $G_2''(AlN)$ for $a = \infty \equiv H_2'(l)$.

269. *Properties of the Groups* $G_2''(AlN)$ and $G_1''(AlS)$. There are ∞^3 conics touching the line l at the point A; and there are ∞^1 circles touching l at A. Each one of these circles is the circle of curvature of a system of ∞^2 conics touching l at A. The group $G_2''(AlN)$, for which a is constant, transforms into itself and thus leaves invariant the net N of ∞^2 conics whose common radius of curvature is $2a$. It contains the group of elations $H_1'(Al)$. .

The group $G_1''(AlS)$ whose parameter is t and for which the law of combination of parameters is $t_2 = t + t_1$, is isomorphic with the parabolic group $G_1'(A)$ of one-dimensional transformations. Its properties are therefore known and need not be restated in detail.

§ 8. Groups of Perspective Collineations.

270. In § 7 we investigated the fundamental groups of collineations of types IV and V, viz.: the one-parameter groups $H_1(A,l)$ and $H_1'(Al)$. In the present § we wish to determine all varieties of groups which contain only perspective collineations. We know from the results of Arts. 123, 228, that there are ∞^5 perspective collineations in the plane and that these do not form a group. We also know that the system of ∞^3 collineations of type IV which have a common axis (or common vertex) do form a group. We have also learned, Art. 124, that there are ∞^4 elations in the plane and these do not form a group; and a number of similar questions have been settled.

But in the present § we wish especially to show how the normal form of the collineation S and S' may be used to develop a complete theory of perspective collineations. We shall also make free use of geometric methods. Perspective collineations are a special kind, and for this reason they are specially fitted to illustrate the various methods that may be employed. We shall therefore ignore for the most part the results already obtained for types IV, V and proceed to investigate these types *de novo;* in so doing we shall sacrifice brevity for the sake of ample illustration. The results obtained will serve as a check upon the methods of §§ 4 and 5.

A. GROUPS OF TYPE V.

271. *Two-parameter Group* $H_2'(l)$. Let us take two elations, designated by $S'(Al)$ and $S_1'(A_1l)$ having the same axis l and their vertices A and A_1 on l, Fig. 27(a), and determine the character of their resultant. Since S' and S_1' both leave invariant every point on l, their resultant also leaves invariant every point on l and is therefore a perspective collineation. Let t and t_1 be the characteristic constants of S' and S_1' respectively, and let us consider the effect of S' and

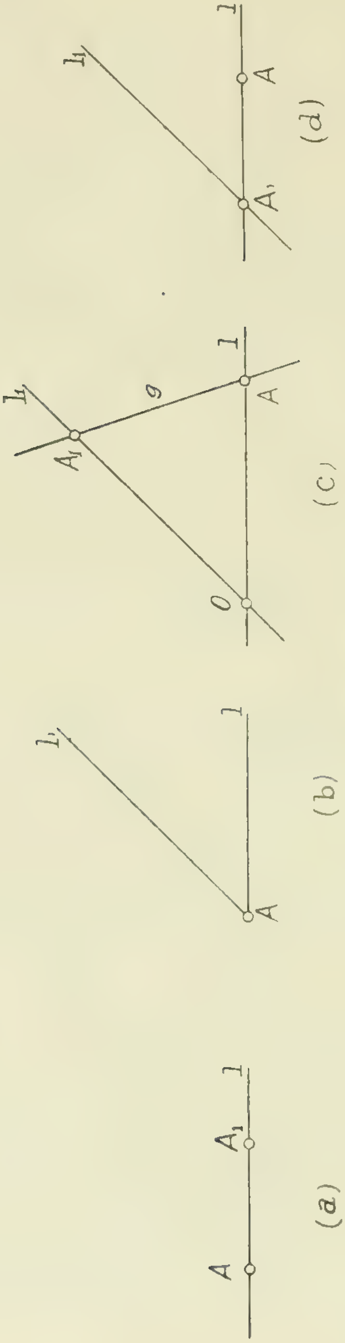
A
A_1
1
(a)
A
l_1
1
(b)
O
A_1
l_1
g
A
1
(c)
A_1
A
l_1
1
(d)

FIG. 27.

S_1' on a line g parallel to l. S' transforms g to g_1 and S_1' transforms g_1 to g_2, both parallel to l. Let the perpendicular distances of g, g_1, and g_2 from l be respectively p, p_1, and p_2. By the equations of Art. 145, we have

$$\frac{1}{p_1} - \frac{1}{p} = t \text{ and } \frac{1}{p_2} - \frac{1}{p_1} = t_1.$$

Eliminating p_1 we get

$$\frac{1}{p_2} - \frac{1}{p} = t + t_1 = t_2.$$

Let us also consider the effect of S' and S_1' on a pencil of rays through O, any point on l. The pencil through O undergoes one-dimensional parabolic transformations due to S' and S_1' which are expressed, Art. 145, by the equations

$$\cot\phi_1 - \cot\phi = dt \text{ and } \cot\phi_2 - \cot\phi_1 = d_1t_1,$$

where $d = 0A$ and $d_1 = 0A_1$. Eliminating $\cot\phi_1$ from the equations we have

$$\cot\phi_2 - \cot\phi = dt + d_1t_1 = d_2t_2,$$

which shows that the resultant one-dimensional transformation in the pencil through O is also parabolic. Hence the resultant of S' and S_1' is an elation $S_2'(A_2l)$ having its vertex at some point A_2 on l. The characteristic constant and vertex of $S_2'(A_2l)$ are given by the equations

$$t_2 = t + t_1 \text{ and } d_2t_2 = dt + d_1t_1. \qquad (64)$$

The first group property is therefore established for the set of ∞^2 elations having a common axis l. This set is made up of ∞^1 one-parameter groups $H_1'(Al)$, one for each point on l. Since the inverse of every collineation in one of these groups is likewise in the same group, it is also in the set made up of these groups. Hence the set of ∞^2 elations having the same axis is a group of two-parameters, t and d, designated by $H_2'(l)$.

272. *Analytic Proof of* $H_2'(l)$. Let the axis of the elation l be taken as the x-axis and let the origin be some point O on l. The equations of S' are found by putting $B = 0\, c = 1c' = 0$

$c_1 = 0$ and $c_1' = 1$ in equation as chapter II, the normal form of type V. We thus get

$$S' : \quad x_1 = \frac{x + Aty}{1 + ty} \text{ and } y_1 = \frac{y}{1 + ty} . \tag{65}$$

S_1' is of the same form with different A and t.

$$S_1' : \quad x_2 = \frac{x_1 + A_1 t_1 y_1}{1 + t_1 y_1} \text{ and } y_2 = \frac{y_1}{1 + t_1 y_1} . \tag{65'}$$

Eliminating x_1 and y_1 we get

$$S_2' : \quad x_2 = \frac{x + A_2 t_2 y}{1 + t_2 y} \text{ and } y_2 = \frac{y}{1 + t_2 y} , \tag{65''}$$

where $t_2 = t + t_1$ and $A_2 t_2 = At + A_1 t_1$. The first group property is thus established analytically, the second is shown to exist by solving (65) for x and y; and again we see the existence of the group $H_2'(l)$.

273. *The Two-parameter Group $H_2'(A)$.* Let us consider two elations, S' and S_1', having the same vertex A and different axes through A. Fig. 28(b). Every line through A is invariant under both S' and S_1' and hence is invariant under their resultant. Along each invariant line through A both S' and S_1' set up one-dimensional parabolic transformations with a common invariant point A. The resultant along each invariant line is therefore parabolic and hence the resultant of S' and S_1' is again an elation with vertex at A. Thus the first group property is proved. Since the inverse of each elation is in the same one-parameter group, the second group property follows and we have established the existence of the two-parameter group of elations having a common vertex. The convenient symbol for the group is $H_2'(A)$.

274. *Resultant of Any Two Elations.* Let $S'(Al)$ and $S_1'(A_1 l_1)$ be any two elations whose invariant figures are in the most general position with respect to each other. Let l and l_1 intersect in the point O and let g be the line joining A and A', Fig. 27(c). Both S' and S_1' leave invariant O and g, hence their resultant leaves both O and g invariant. Along the line g we have two one-dimensional parabolic transforma-

tions with different invariant points. Their resultant is (Chap. I, Art. 22) usually loxodromic with two invariant points, say B and C; it may however be parabolic with one invariant point, B; or it may be identical leaving all points on g invariant. In the first case if there are two invariant points B and C on g, then the resultant of S' and S_1' is of type I, leaving invariant the triangle (OBC); if in the second case there is only one invariant point B on g, then the resultant of S' and S_1' is of type II, leaving invariant the figure (OBg); if in the third case every point in g is invariant, the resultant of S' and S_1' is of type IV, having O for its vertex and g for its axis. Hence the resultant of two elations in the most general position is of type I, type II or type IV.

275. *Two Elations Whose Resultant is of Type III.* In the above configuration let O, the intersection of l and l_1, coincide with A_1 the vertex of S_1', Fig. 27(d). The resultant of S' and S_1' now leaves invariant the point A_1 and the line l. Along the line l the two component one-dimensional transformations are respectively parabolic, with invariant point at A_1, and identical; the resultant along l is therefore parabolic with invariant point at A_1. The two component one-dimensional transformations of the pencil through A_1 are respectively parabolic and identical; their resultant is therefore parabolic, having l as the invariant line. The resultant of S' and S_1' leaves invariant the lineal element A_1l and produces one-dimensional parabolic transformations along l and through A_1 and is therefore of type III.

276. *Analytic Proofs of Articles* (273 275). The results of the last three articles may be deduced analytically. We shall only outline the proof, leaving the details to the reader. Starting with the configuration of Art. 273 let O be the origin and l and l_1 the x- and y-axis respectively. The normal forms of S' and S_1' reduce to the following:

$$S' \ : \quad x_1 = \frac{x + Aty}{1 + ty} \text{ and } y_1 = \frac{y}{1 + ty} \ ; \tag{66}$$

$$S_1' : \quad x_2 = \frac{x_1}{1+t_1x_1} \text{ and } y_1 = \frac{B_1t_1+y_1}{1+t_1x_1}. \tag{66'}$$

The resultant will be of type I in general; it will reduce to type III, if $B=0$; and to type V if both $A=0$ and $B_1=0$, thus proving the existence of the group $H_2'(A)$.

THEOREM 49. The ∞^4 elations of the plane do not form a group: there are three varieties of groups of elations, viz.: $H_1'(Al)$ $H_2'(l)$ and $H_2'(A)$.

B. GROUPS OF TYPE IV.

277. *Resultant of S and S_1.* We shall now take up the problem of determining the character of the resultant of any two collineations of type IV, and all continuous groups of such collineations. We shall begin with the analytic determination of the resultant of two perspective collineations having the same axis.

Let S and S_1 be two perspective collineations having the same axis but not the same vertex. Fig. 28(a). Let us choose the common axis of S and S_1 as the x-axis, and let the vertices of S and S_1 be any two points in the plane. Starting with the normal form of type I we put $k=1$, $A=0$, $B=0$, $B'=0$ in equations (12), chap. II. We thus get, after expanding and reducing,

$$S : \; x_1 = \frac{x + \frac{A''}{B''}(k_1-1)y}{1+\left(\frac{k_1-1}{B''}\right)y}, \qquad y_1 = \frac{k_1 y}{1+\left(\frac{k_1-1}{B''}\right)y}.$$

The vertex is the point whose coordinates are $(A''B'')$ and k' is the characteristic cross-ratio of the perspective collineation S. Dropping primes and double primes, since they are no longer necessary, we can put (67) in the form,

$$\frac{x_1}{y_1} = \frac{x}{ky} + \frac{A}{B}\left(\frac{k-1}{k}\right), \quad \frac{1}{y_1} = \frac{1}{ky} - \frac{1}{B}\left(\frac{k-1}{k}\right). \tag{67}$$

Writing S_1 in the same form with different vertex (A_1B_1) and different cross-ratio k_1, we have

$$\frac{x_2}{y_2} = \frac{x_1}{k_1\,y_1} + \frac{A_1}{B_1}\left(\frac{k-1}{k}\right), \quad \frac{1}{y_2} = \frac{1}{k_1\,y_1} - \frac{1}{B_1}\left(\frac{k_1-1}{k_1}\right). \tag{67'}$$

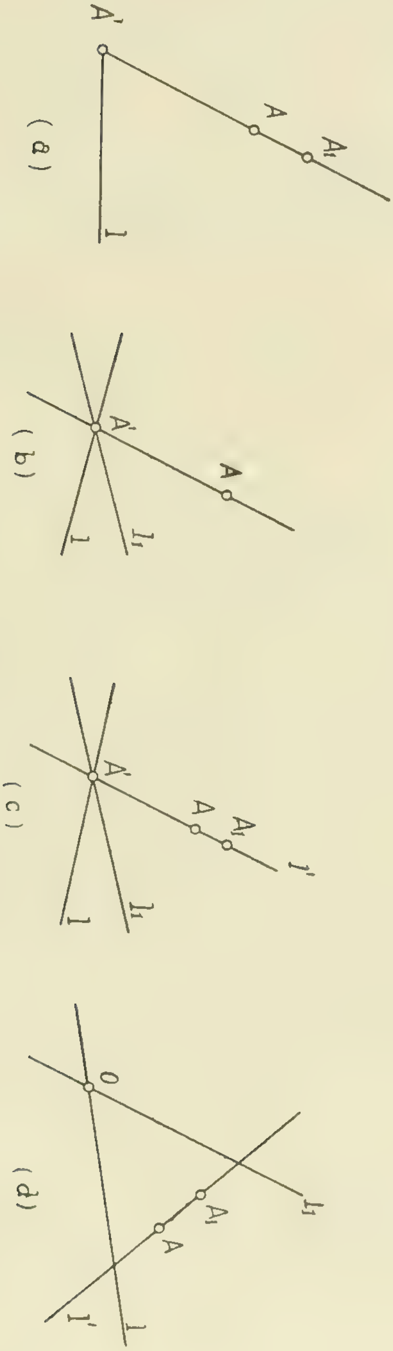
(a)
A'
A
A₁
1
(b)
A'
A
1
1₁
(c)
A'
A
A₁
1'
1
1₁
(d)
O
A₁
A
1₁
1'
1

FIG. 28.

Eliminating x_1 and y_1 from these sets of equations, we get the resultant of S and S_1 in the same form as S, viz.:

$$\begin{aligned} \frac{x_2}{y_2} &= \frac{x}{kk_1 y} + \frac{A}{B}\left(\frac{k-1}{kk_1}\right) + \frac{A_1}{B_1}\left(\frac{k_1-1}{k_1}\right) = \frac{x}{k_2 y} + \frac{A_2}{B_2}\left(\frac{k_2-1}{k_2}\right), \\ \frac{1}{y_2} &= \frac{1}{kk_1 y}\left(\frac{k_1-1}{B_1 kk_1} + \frac{k_1-1}{B_1 kk_1}\right) = \frac{1}{k_2 y} - \frac{k_2-1}{B_2 k_2}. \end{aligned} \tag{67''}$$

Comparing coefficients of like terms in the two forms of (32″) we have three equations, as follows:

$$\begin{gathered} k_2 = k\,k_1,\ \frac{A_2}{B_2}\left(\frac{k_2-1}{k_2}\right) = \frac{A}{B}\left(\frac{k-1}{kk_1}\right) + \frac{A_1}{B_1}\left(\frac{k_1-1}{k_1}\right), \\ \frac{k_1-1}{B_2\,k_2} = \frac{k-1}{Bkk_2} + \frac{k_1-1}{B_1\,k_1}. \end{gathered} \tag{68}$$

The fact that (67″) is of the same form as (67), shows that the resultant of S and S_1 is a perspective collineation S_2 having the axis coinciding with the axis of S and S_1. Equations (68) give us the values k_2, A_2 and B_2, in terms of k, k_1, A, B, A_1 B_1. We can now state the result:

The resultant of two perspective collineations having the same axis, but different vertices, is a perspective collineation with the same axis; the cross-ratio of the resultant is the product of the cross-ratios of the components.

278. *The Group $H_3(l)$.* Equations (67) contain three parameters k, A, B; hence there are ∞^3 perspective collineations having a common axis. It has just been shown that this set of perspective collineations has the first group property, viz.: the resultant of any two of the set is one of the same set. We can also show that the set has the second group property, viz.: the inverse of one of the set is also in the set. If (67) be solved for $\frac{x}{y}$ and $\frac{1}{y}$ we get the inverse:

$$\frac{x}{y} = k\frac{x_1}{y_1} + \frac{A}{B}(k-1), \quad \frac{1}{y} = \frac{k}{y_1} - \frac{k-1}{B}. \tag{67'''}$$

This is of the same form as (67) except that k is changed into $1/k$; hence the inverse of every collineation in the set is also in the set and the set is a group. The symbol for the group is $H_2(l)$.

All perspective collineations having a common axis form a three-parameter group.

279. *The Group* $H_2(ll')$. We have already seen that the cross-ratio of the resultant of any two collineations in $H_2(l)$ is equal to the product of the cross-ratios of the components; thus $k_2 = kk_1$. We next inquire into the position of the vertex of the resultant, whose coordinates are (A_2, B_2).

From equations (68) we find

$$A_2 = \frac{AB_1(k-1) + A_1Bk(k_1-1)}{B_1(k-1) + Bk(k_1-1)}, \quad B_2 = \frac{BB_1(kk_1-1)}{B_1(k-1) + Bk(k_1-1)}. \tag{69}$$

It is easy to verify the following equation,

$$\begin{vmatrix} A & B & 1 \\ A_1 & B_1 & 1 \\ A_2 & B_2 & 1 \end{vmatrix} = 0. \tag{70}$$

Hence the point (A_2, B_2) is collinear with (A, B) and (A_1, B_1). From this fact we infer that all collineations having the same axis and whose vertices are collinear form a group. This group is designated by $H_2(ll')$, l' being the line on which the vertices lie.

The group $H_3(l)$ contains ∞^2 two-parameter sub-groups $H_2(ll')$, one for each line l' of the plane; but two such sub-groups contain one one-parameter group $H_1(A, l)$ in common, A being the point of intersection of l' and l_1'.

280. *The Groups* $H_3(A)$ *and* $H_2(AA')$. It may be shown in a manner similar to Art. (278) that all perspective collineations having the same vertex A form a three-parameter group $H_3(A)$, and that all such collineations having their axes concurrent at A' form a two-parameter group $H_2(AA')$. The same thing may be shown geometrically as follows: Let $S(A, l)$ and $S_1(A, l_1)$ be two perspective collineations having the common vertex A. Fig. 28(b). Both leave invariant all lines through A; hence their resultant also leaves invariant the pencil through A and is also a perspective collineation. Both also leave invariant A', the intersection of l and l_1; hence the resultant is of type IV and its axis, l_2, goes

through A'. This proves the existence of both groups $H_3(A)$ and $H_2(AA')$.

All perspective collineations having a common vertex A form a group $H_3(A)$; all of these whose axes are concurrent form a group $H_2(AA')$; all which have both vertices and axes in common form a group $H_1(A_1l)$.

281. *No Group* $H_3(A'l')$. We wish now to use the analytic method to disprove the existence of a group which might be mistakenly inferred from the geometrical point of view. It is plausible to infer that the ∞^3 perspective collineations whose axes are concurrent through A' and whose vertices are on l', collinear with A', form a group, Fig. 28 (*c*). To test this we have only to write down two such collineations and form their resultant. The group property is proved or disproved according as the resultant is a perspective or a non-perspective collineation.

Taking A' for the origin and l' for the x-axis, S reduces to the form

$$S\ :\ x_1 = \frac{x}{1+p\,\frac{1-k}{B}\,x - \frac{(1-k)}{B}\,y}\,,\quad y_1 = \frac{p\,(1-k)\,x + ky}{1+p\,\frac{1-k}{B}\,x - \frac{1-k}{B}\,y}\,, \tag{71}$$

where $p = \frac{c'}{c} = tan\ \phi$, the slope of the axis of S.

A second collineation S_1 of the same set is

$$S\ :\ x_2 = \frac{x_1}{1+p_1\,\frac{1-k_1}{B}\,x_1 - \frac{(1-k_1)}{B_1}\,y_1}\,,\ y_2 = \frac{p_1\,(1-k_1)\,x_1 + k_1 y_1}{1+p_1\,\frac{1-k_1}{B_1}\,x_1 - \frac{1-k_1}{\quad}}\,. \tag{71'}$$

The resultant is

$$x_2 = \frac{x}{1+\left\{p\,\frac{1-k}{B} + p_1\,\frac{1-k_1}{B_1} - p\,\frac{(1-k)\ (1-k_1)}{B_1}\right\}x - \left\{\frac{1-k}{B} + \frac{(1-k_1)}{B_1}\right\}y}\,. \tag{71''}$$

$$y_2 = \frac{\left\{p_1\,(1-k_1) + p\,k_1\,(1-k)\right\}x + k\,k_1\,y}{1+\left\{p\,\frac{1-k}{B} + p\,\frac{1-k_1}{B_1} - p\,\frac{(1-k)\ (1-k_1)}{B_1}\right\}x - \left\{\frac{1-k}{B} + \frac{k\,(1-k_1)}{B_1}\right\}y}\,.$$

Comparing coefficients of x and y in components and resultants we have

$$k_2 = k k_1,\ p_2(1-k_2) = p_1(1-k_1) + p k_1(1-k), \qquad (72)$$

$$\frac{1-k_2}{B_2} = \frac{1-k}{B} + \frac{k(1-k)}{B},\ p_2\frac{1-k_2}{B_2} = p\frac{1-k}{B} + p_1\frac{1-k}{B_1} - p\frac{(1-k)(1-k_1)}{B_1}.$$

We have here four independent equations involving the three quantities k_2, B_2, p_2. Hence the resultant is not of type IV and the set of perspective collineations to which S and S_1 belong does not form a group. Equations (71″) represent a collineation of type II leaving invariant the origin, the y-axis, the point $(0, B_2)$ on the y-axis, and a line l_2 through the origin, but no other point on l_2.

Equations (72) show that no group of the kind $H_3(A'l')$ exists; but if $B_1 = B$ in (71′) and (71″) they show the existence of the group $H_2(AA')$; if $p_1 = p$ in the same equations they show again the existence of the group $H_2(ll')$.

282. *Resultant of Any Two Perspective Collineations.* Let S and S_1 be two collineations of type IV whose axes l and l_1 intersect in O and whose vertices A and A_1 lie on a line l'. Fig. 28(d). The resultant of S and S_1 leaves O and l' invariant. Along l' the two one-dimensional transformations due to S and S_1 result generally in a one-dimensional transformation with two invariant points A and A'. Hence the resultant of S and S_1 leaves invariant the triangle (OAA') and is of type I. The non-existence of the group property is thus proved for the ∞^5 perspective collineations.

THEOREM 50. The ∞^5 perspective collineations of the plane do not form a group; there are five varieties of groups of type IV, viz.: $H_3(A)$, $H_3(l)$, $H_2(AA')$, $H_2(ll')$, $H_1(Al)$.

283. *Dualistic and Self-dualistic Groups.* From the dualistic character of a collineation, we infer that its fundamental invariant figure is self-dualistic. When there exists a group of collineations, leaving a certain figure invariant, there must also exist a second group leaving invariant a figure dualistic

to the first. Two such groups are said to be dualistic to one another; if the invariant figure of the group is a self-dualistic figure, the group is called a self-dualistic group.

As examples, we may cite the following pairs of dualistic groups: $H_3(A)$ and $H_3(l)$, $H_2(ll')$ and $H_2(AA')$, $H_2'(A)$ and $H_1'(l)$; the one-parameter groups $H_1(A, l)$ and $H_1'(Al)$ are self-dualistic groups.

This principle holds for all collineations of whatever kind, and hereafter, when the existence of a certain group has been proved, it will be assumed, without further proof, that the dualistic group also exists and has properties dualistic to the first.

284. *Resume of Perspective Collineations.* In the following list, the structure of all groups of perspective collineations is indicated. A dash above a letter indicates that the line or point thus marked takes on different positions in the invariant figure of the group. Thus $H_2'(l) = \infty H_1'(\overline{A}l)$ indicates that the point A takes all positions on the line l. Dualistic groups are bracketed together, and self-dualistic groups are bracketed alone.

$$\{ H_1'(Al) \},$$

$$\{ H_1(A, l) \},$$

$$\left\{\begin{array}{ll} H_2'(l) & = \infty^1 H_1'(\overline{A}l), \\ H_2'(A) & = \infty^1 H_1'(Al), \end{array}\right.$$

$$\left\{\begin{array}{ll} H_2(ll') & = \infty^1 H_1(\overline{A}, l) + H_1'(Bl), \\ H_2(AA') & = \infty^1 H_1(A, \overline{l}) + H_1'(Al'), \end{array}\right.$$

$$\left\{\begin{array}{ll} H_3(l) & = \infty^2 H_1(\overline{A}, l) + H_2'(l), \\ H_3(A) & = \infty^2 H_1(A, l) + H_2'(A). \end{array}\right.$$

§9. Groups of Types I, II and III.

In the last § we found the complete list of varieties of subgroups of G_8 that contain only collineations of types IV and V. In the present § we take up the problem of finding the complete list for types I, II and III. In § 4 we found all varieties of subgroups of G_1 that are defined by linear and quadratic relations on the elements of M; and it was there expressly assumed that the elements of M were not subject to any of the restrictions that would make the characteristic equations of M have multiple roots. Hence the groups of the lists given in Theorem 12 are all of type I. In § 5 we found all varieties of subgroups of types II and III that are defined by linear relations in the elements of M. But it does not follow that these are the complete lists of groups of these types, for each of these groups may have subgroups defined by some additional condition and still of their respective types. For example we found in § 7 that the group $G_2(AA'A'')$ has ∞^1 subgroups of type I each characterized by a constant value of r. We shall find that each of the groups listed in Theorem 12, except $G_1(AA'A''K)$, has one or more subgroups of type I; and that each group listed in Theorem 22 has ∞^1 subgroups of type II.

A. GROUPS OF TYPE I.

285. *Three Classes of Groups of Type I.* We wish to make a rational classification of the groups of type I. In so doing we shall find three distinct classes of such groups, viz.: (1) Those containing collineations in which the cross-ratio parameters k and k' are independent of each other; groups of this kind are made up of two-parameter groups of the kind $G_2(AA'A'')$; (2) another class containing collineations in which the cross-ratio parameters satisfy the relation $k' = k^r$ for a constant value of r; the groups of this class are made up of one-parameter groups of the kind $G_1(AA'A'')_r$; (3) a class of groups for which $r = -1, 2, 1/2$, these are made up

of one-parameter groups whose path-curves are conics. Strictly speaking the third class is only a special case of the second, but we shall see that the special case is so important that these groups should be classified by themselves.

286. *Groups of the First Class.* One list of groups of the first class is already complete. It consists of G_8 and the subgroups of G_8 defined by one or more sets of linear relations on the elements of M. These are by Art. (209) $G_6(A)$, $G_6(l)$, $G_5(Al)$, $G_4(AA')$, $G_4(ll')$, $G_4(A, l)$, $G_3(AA'l)$, $G_2(AA'A'')$. We include here the general projective group G_8, since it can be built up of the ∞^6 two-parameter groups $G_2(AA'A'')$ of the plane and cannot be included in any of the other classes of type I.

287. *Groups of the Second Class.* We now proceed to the determination of groups of another variety, viz.: those in which every collineation is characterized by the same value of r. It has already been shown, Art. 249, that the two-parameter group $G_2(AA'A'')$ breaks up into ∞^1 one-parameter subgroups, each subgroup being characterized by a constant value of r. If we take all types of groups of the first class and in these set $k' = k^r$ and keep r constant and let the other parameters vary, we will sometimes find subgroups by this process and sometimes not. The groups G_8; $G_6(l)$, $G_6(A)$; $G_5(Al)$; $G_4(A, l'')$, $G_4 AA')$, $G_4(ll')$; $G_3(AA'l')$ must each be examined separately.

Every group of the first class which has the double k-relations $k_2 = k k_1$ and $k_2' = k' k_1'$ may be broken up into subgroups of the second class. To show this let $k' = k^r$ and $k_1' = k_1^r$, where r is any constant. Then $k_2' = k' k_1' = k^r k_1^r (k k_1)^r = k_2^r$. Hence the two conditions $k_2 = k k_1$ and $k_2' = k' k_1'$ reduce to a single one and all the collineations in the group of the first class satisfying the relations $k' = k^r$ form a subgroup. There will be one such subgroup for every value of r.

It was shown in Art. 234 that there are five groups of the first class which have the double k-relations $k_2 = k k_1$ and

$k_2' = k'k_1'$ and which therefore break up into groups of the second class, viz.: $G_5(Al)$, $G_4(AA')$, $G_4(ll')$, $G_3(AA'l)$ and $G_1(AA'A'')$. We shall designate these groups of the second class by $G_4(Al)_r$, $G_3(AA')_r$, $G_2(ll')_r$, $G_2(AA'l)_r$, $G_1(AA'A'')_r$.

It is easy to show that the remaining subgroups of G_5 of the first class do not break up into subgroups in the above manner. The groups $G_6(A)$, $G_6(l)$ and $G_4(A, l)$ have the single k-relation $k_2 k_2' = k k' k_1 k_1'$; if we put $k' = k^r$ and $k_1' = k_1^r$ we do not find $k_2' = k_2^r$ for all values of r. Hence these groups do not break up into subgroups of the second class. There is, however, one exceptional value of r for which those three groups have one subgroup each; this exceptional case is considered later, Art. 289.

The six critical values of r, viz.: $r = 1, 0, \infty, -1, 2, \frac{1}{2}$, are to be excepted for each of these groups of the second class. For $r = 1, 0, \infty$ the collineations are all perspective collineations and the groups corresponding to these values of r are groups of perspective collineations. For $r = -1, 2, \frac{1}{2}$ the path-curves are conics and the resulting groups belong to the third class to be considered below.

THEOREM 51. There are five, and only five, varieties of groups of the second class, viz.: $G_1(ABC)_r$, $G_2(ABl')_r$, $G_3(AB)_r$, $G_3(ll')_r$, $G_4(Al)_r$.

288. *Groups Whose Path-curves are Conics.* We come now to the consideration of the groups of collineations of type I which are made up of one-parameter groups whose path-curves are conics. These groups are in many instances only special cases of groups of the second class when $r = -1, 2, \frac{1}{2}$; but we shall also find many groups which are not special cases of the above. These latter cases are usually of great interest.

There is only one variety of one-parameter group of this kind; in this case the two common tangents to the conic and the common chord of contact form the invariant triangle.

We shall denote the configuration, see Fig. 26, by $(AA'A''K)$ and the group is $G_1(AA'A''K)$.

289. *Subgroups of* $G_4(A, l'')$, $G_6(A)$, $G_6(l)$ *when* $r = -1$. The three groups $G_6(A)$, $G_6(l)$ and $G_4(A, l)$ have the single k-relation $k_2k_2' = kk'k_1k_1'$, but not the double k-relations. Hence these groups do not break up into groups of the second class for all values of r. But for $r = 1$ we do find a subgroup in each of the above groups of the first class. Let $k' = k^r$ and $k_1' = k_1^r$ in the relation $k_2k_2' = kk_1k'k_1'$, then, we have $k_2k_2' = (kk_1)^{1+r}$. If now $r = -1$, then $k_2' = k_2^{-1}$; *i. e.*, the two cross-ratios in the resultant have the same relation to each other as in the components. Hence for $r = -1$ we find the three groups $G_3(A, l'')_{r=-1}$, $G_5(A)_{r=-1}$, and $G_5(l)_{r=-1}$.

THEOREM 52. The groups $G_4(A, l'')$, $G_6(A)$ and $G_6(l)$ each contain one and only one subgroup for a constant value of r, viz.: when $r = -1$, $G_5(A)_{r=-1}$, $G_5(l)_{r=-1}$.

290. *The Structure of* $G_4(Al)\, r = 2$. The group $G_4(Al)$ $r = 2$ breaks up into subgroups in several different ways, and accordingly its structure is peculiar and worthy of attention.

There are ∞^3 distinct conics touching l at A. This system of conics is composed of ∞^1 nets N, such that each net contains ∞^2 conics having contact of the second order at A, and hence all conics of a net N have a common circle of curvature at A. Each of the circles touching l at A is the circle of curvature of such a net.

Each net N is composed of ∞^1 pencils S such that the conics of each pencil have contact of the third order at A. The polar of the point at infinity on l with respect to the conics of the pencil S is a line l' through A, the line of centers of S. Each line through A is the line of centers of one of the pencils of the net N. Since there are ∞^1 nets N, each line through A is the line of centers of ∞^1 pencils S. The system of conics touching l at A is, therefore, composed of ∞^2 pencils S.

Now we know that the group $G_4(Al)_{r=2}$ contains the following varieties of subgroups, $G_3(AA')_{r=2}$, $G_3(ll')_{r=2}$, $G_2(AA'l')_{r=2}$. We must examine the three remaining configurations, viz.: (AlK), (AlS) and (AlN), and see if these are invariant under certain groups.

291. *The Group* $G_2(AlK)$. Let us take any conic K of the system touching l at A. We can construct ∞^1 triangles, Fig. 29, having one vertex at A, another at A' on l, and a third vertex A'' on the conic, such that AA' and $A'A''$ are tangents to K and AA'' is the chord of contact. Belonging to each of these triangles is a one-parameter group whose path-curves are conics all touching AA' at A. Each of these groups has K among its invariant path-curves. Hence these ∞^2 collineations leaving invariant the configuration (AlK) form a group of two parameters, $G_2(AlK)$, the two parameters being the cross-ratios k along l and the position of the point A'' on K. The group $G_4(Al)_{r=2}$ contains ∞^3 such subgroups, one for each conic in the system touching l at A.

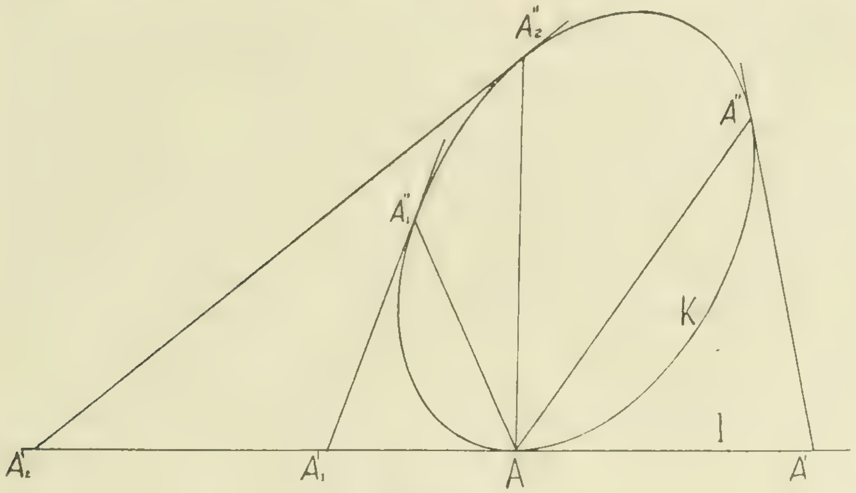

FIG. 29.

THEOREM 53. All collineations leaving invariant the configuration consisting of a conic, one of its tangents, and the point of contact, form a two-parameter group $G_2(AlK)$.

292. *The Group* $G_3(AlS)$. Let us select from the system of conics touching l at A any pencil S such that all conics of S have contact of the third order with each other at A.

One property of this pencil of conics is that the polars of a point on l with respect to the pencil S coincide, and this common polar passes through A. A pencil of conics having contact of the third order is projectively transformed into a pencil of the same kind. We can construct ∞^2 triangles having one side AA' along l, one vertex at A, and one vertex at A'' on one of these conics, so that $A'A''$ is a tangent to the curve at A'', and AA'' is the chord of contact. Belonging to each of these triangles is a one-parameter group $G_1(AA'A'')_{r=2}$ such that one conic of the pencil S is included in its family of path-curves. Every collineation in such a one-parameter group leaves invariant one of the conics of S and interchanges the other conics of S; hence, it leaves invariant the pencil S as a whole and the lineal element Al. The aggregate of all collineations, leaving S invariant, forms a three-parameter group $G_3(AlS)$.

Since the group $G_3(AlS)$ leaves invariant the lineal element Al, the pencil of conics S having contact of the third order at A, we see that $G_3(AlS)$ can be built up out of ∞^1 two-parameter groups $G_2(AlK)$, one for each conic in S. Again, since the polars of any point A' on l with respect to each conic of S coincide in l', a line through A, we see that $G_3(AlS)$ is composed of ∞^1 subgroups $G_2(A'Al')_{r=2}$.

THEOREM 54. All collineations leaving invariant the configuration consisting of a pencil of conics S having contact of the third order with each other, and their common point A and line l, form a three-parameter group $G_3(AlS)$.

293. *Analytic Determination of* $G_3(AlS)$ *and* $G_2(AlK)$. The group $G_5(Al)$ is reducible, hence the point A may be taken as one vertex of the triangle of reference, as $(0, 0, 1)$, and the line l as an adjacent side, as $y = 0$ of the triangle of reference. We may therefore put $A = B = B' = 0$ in the normal form of T, and thus get the reduced normal form

of T for this group. Putting $A=B=B'=0$ and $A_1=B_1=B_1'=0$ in equations I IX of Art. 179 and dividing the others by IX we get the five following equations of the conditions for the group $G_5(Al)$:

$$\text{I. } k_2=kk_1\,;\quad \text{II. } k_2'=k'k_1'\,;\quad \text{III. } \frac{k_2-1}{A_2'}=\frac{k-1}{A'}+\frac{k(k_1-1)}{A_1'}\,;$$
$$\text{IV. } (k_2'-k_2)\frac{A_2''}{B_2''}=k_1(k'-k)\frac{A''}{B''}+k'(k_1'-k_1)\frac{A_1''}{B_1''}\,;\qquad (73)$$
$$\text{V. } \frac{k_2'-1}{B_2''}-\frac{k_2-1}{A_2'}\frac{A_2''}{B_2''}=\frac{k'-1}{B''}-\frac{k-1}{A'}\frac{A''}{B''}+\frac{(k'-k)(k_1-1)}{A_1'}\frac{A''}{B''}+$$
$$\frac{k'(k_1'-1)}{B_1''}-\frac{k'(k_1-1)}{A_1'}\frac{A_1''}{B_1''}\,.$$

Making $k'=k^2$ and $k_1'=k_1^2$, then $k_2'=k_2^2$; equations I–V then reduce to the following equations of condition of group $G_4(Al)_{r=2}$:

$$\text{I. } k_2=kk_1\,;\quad \text{III. } \frac{k_2-1}{A'_2}=\frac{k-1}{A'}+\frac{k(k_1-1)}{A_1'}\,;$$
$$\text{IV. } (k_2-1)\frac{A_2''}{B_2''}=(k-1)\frac{A'}{B''}+k(k_1-1)\frac{A_1''}{B_1''}\,;\qquad (74)$$
$$\text{V. } \frac{k_2^2-1}{B_2''}-\frac{k_2-1}{A_2'}\frac{A_2''}{B_2''}=\frac{k^2-1}{B'}-\frac{k-1}{A'}\frac{A''}{B''}+\frac{k(k-1)\ (k_1-1)}{A_1'}\frac{A''}{B''}+$$
$$\frac{k^2(k_1^2-1)}{B_1''}-\frac{k^2(k_1-1)}{A_1'}\frac{A_1''}{B_1''}\,.$$

If in these equations we make $A_1'=A'=A_1'$ (III) disappears and we get the equations of the group $G_3(AA')_{r=2}$. If on the other hand we make $\frac{A_1''}{B_1''}=\frac{A''}{B''}=\frac{A_2''}{B_2''}$, then (IV) disappears and we have left the equations of the group $G_3(ll')_{r=2}$. If both assumptions are made simultaneously, equations III and IV both disappear and we have left the equations of the group $G_2(AA'l)_{r=2}$. If $A_1'=A'=A_2'$, and $A_2''=A''=A_2''$, and $B_1''=B''=B_2''$, then equations III, IV and V all disappear and we have left only $k_2=kk_1$ which is the equation of condition of the group $G_1(AAA')_{r=2}$. Hence we see that the group $G_4(Al)_{r=2}$ contains the following varieties of subgroups: $G_3(AA')_{r=2}$, $G_3(ll')_{r=2}$, $G_2(AA'l)_{r=2}$, and $G_1(AAA')_{r=2}$. We shall go on to show that $G_4(Al)_{r=2}$ also contains other varieties of of subgroups.

294. *The Group* $G_3(AlS)_{r=2}$. If we set $\frac{A''}{B''}=\frac{2a}{A'}$ and $\frac{A_1''}{B_1''}=\frac{2a}{A_1'}$, (or in more general form $\frac{A''}{B''}+h=\frac{2a}{A'}$ and $\frac{A_1''}{B_1}+h=\frac{2a}{A_1'}$), in the conditional equations of $G_4(Al)_{r=2}$, we have $\frac{A_2''}{B_2''}=\frac{2a}{A_2''}$ (or more generally $\frac{A_2''}{B_2''}+h=\frac{2a}{A_2''}$), and IV reduces to III. We thus have left the equations of a three-parameter subgroup of $G_4(Al)_{r=2}$. It is not difficult to see that this three-parameter group leaves invariant a pencil of conics S touching l at A and should therefore be designated by $G_3(AlS)_{r=2}$. The equations of such a system of conics may be written

$$x^2+2hxy+Cy^2=4ay, \tag{75}$$

where C is the parameter of the pencil. The polar of any point A' on l with respect to this pencil of conics is

$$A'x+hA'y=2ay \quad \text{or} \quad \frac{x}{y}+h=\frac{2a}{A'}.$$

Hence the condition $\frac{A''}{B''}+h=\frac{2a}{A'}$, implies that the point $(A'B'')$ is on the polar of the point (A', O) with respect to every conic of the pencil given by equation (52).

It is easy to show by direct substitution that the transformation T which satisfies the above conditions leaves invariant the pencil of conics

$$x^2+2hxy+Cy^2=4ay.$$

The group $G_4(Al)$ contains therefore ∞^2 subgroups $G_3(AlS)_{r=2}$, one for each pencil S touching l at A.

295. *The Group* $G_2(AlK)$. The pencil S contains ∞^1 conics and is invariant under ∞^3 collineations of the group $G_3(AlS)_{r=2}$. If we choose from the pencil S a single conic K, and from the group those collineations which leave K alone invariant, we shall have a subgroup of $G_3(AlS)_{r=2}$.

If in addition to the condition $\frac{A''}{B''}=\frac{2a}{A'}$ we also put $\frac{1}{B''}=\frac{a}{A'}$. (or, more generally, if we put $\frac{A''}{B''}+h=\frac{2a}{A'}$ and $\frac{1}{B''}=\frac{a}{A'_2}+C$) in the three equations of condition of $G_3(AlS)_{r-2}$ these three will reduce to two, viz.: $k_2=kk_1$ and $\frac{k_2-1}{A'_2}=\frac{k-1}{A'}+\frac{k(k_1-1)}{A'_1}$. We thus obtain a two-parameter subgroup of $G_3(AlS)_{r=2}$, which we shall call $G_2(AlK)_{r-2}$. The equation of the conic K is obtained by eliminating A' from the two conditions $\frac{A''}{B''}+h=\frac{2a}{A'}$ and $\frac{1}{B''}=\frac{a}{A'_2}+C$. We find that the point (A'', B'') always lies on the axis $x^2+2hxy+(h^2-4aC)y^2=4ay$, where h, a and C are fixed numbers. The group $G_3(AlS)$ contains ∞^1 subgroups $G_2(AlK)$, one for each conic in S.

296. *Collineations Common to* $G_2(AlK)$ *and* $G_2(AlK')$. Let K and K' be two conics of the system touching l at A. We wish to determine whether the two groups $G_2(AlK)$ and $G_2(AlK')$ have any collineations in common. If K and K' belong to the same pencil S, the two groups have in common the subgroup $G_1''(AlS)$ of type III. If the conics K and K' belong to different pencils S and S' of the same net N, they have three points in common at A and intersect in only one other common point. In this case the groups have no collineations in common. If the conics K and K' do not belong to the same net, they may have double contact at A and A'', and then the two groups have the subgroup $G_1(AA'A''K)$ in common since K and K' both belong to the same pencil K_n. If the conics K and K' intersect in two points other than A, then the groups have no collineations in common.

297. *Collineations Common to* $G_3(AlS)$ *and* $G_3(AlS')$. Each pencil S is the invariant figure of a group $G_3(AlS)$, hence, there are ∞^2 such groups all contained within the group $G_4(Al)_{r-2}$. But since $G_4(Al)_{r-2}$ contains ∞^4 collineations and ∞^2 subgroups $G_3(AlS)$, it follows that two such subgroups as $G_3(AlS)$ and $G_3(AlS')$ must contain certain collineations in common. Let us consider two pencils of conics S and S'

which do not belong to the same net N. A given conic K of the pencil S is cut by each conic of S' in two points other than A. When these two points of intersection of K and K' coincide, the conics K and K' are in contact. Each conic of S has a second contact with one, and only one, conic of S'. The locus of the points of contact of corresponding conics of S and S' is a line l' through A. Therefore, we see that the two three-parameter groups $G_3(AlS)$ and $G_3(AlS')$ contain the same two-parameter group $G_2(AA'l')_{r=2}$.

On the other hand, if the two pencils S and S' belong to the same net N of conics having contact of the second order at A, the conics from S and S' cannot have another contact, and hence two such groups $G_3(AlS)$ and $G_3(AlS')$ have no collineations in common.

In a certain net N there are ∞^1 pencils S, S', S'', etc.; each of these pencils is the invariant pencil of a group $G_3(AlS)$. ∞^1 such groups, no two of which contain a collineation of type I in common, include all collineations of this type in $G_4(Al)_{r=2}$. It is easy to see that the ∞^3 subgroups $G_1(AA'A'')_{r=2}$ in $G_4(Al)_{r=2}$ are all included in the ∞^1 groups $G_3(AlS)$ of the net N. If it be true that all collineations of type I in $G_3(Al)_{r=2}$ are included in the ∞^1 groups $G_3(AlS)$ of the net N, then all collineations of type I in a group $G_3(AlS')$, where S' is a pencil of conics not included in N, are to be found in the ∞^1 groups $G_3(AlS)$ of the net N. In fact, if we take $G_3(AlS')$ in turn with each of the groups $G_3(AlS)$ of the net, we see that $G_3(AlS')$ has a two-parameter subgroup $G_2(ABl')_{r=2}$ in common with each group $G_3(AlS)$ of the net; the common subgroup $G_2(ABl')_{r=2}$ is different for each of the groups $G_3(AlS)$ of the net. In this way it can be shown without difficulty that every collineation of type I in $G_3(AlS')$ is also to be found in the net of groups $G_3(AlS)$.

THEOREM 55. The group $G_4(Al)_{r=2}$ contains ∞^2 subgroups $G_3(AlS)$; two subgroups $G_3(AlS)$ and $G_3(AlS')$ contain no collineation of type I in common when the two pencils are S and S' belong to the same net N: they have a subgroup $G_2(AA'l)_{r=2}$ in common when S and S' belong to different nets. Each group $G_3(AlS)$ contains ∞^1 subgroups $G_2(AlK)$, one for each conic in S.

298. *The Group* $G_3(K)$. Among the groups of the third class must be included the irreducible group $G_3(K)$. It contains, Art. 204, the subgroups $G_2(AlK)$ and $G_1(AA'A''K)$, both of type I and third class. These subgroups have both been found again by other methods.

We have thus found eleven varieties of groups of the third class. Five of these eleven groups, viz.: $G_1(AAA')_r$, $G_2(AA'l)_r$, $G_3(AA')_r$, $G_3(ll)_r$ and $G_4(Al)_r$, where $r=-1$, 2 or $\frac{1}{2}$, are only special cases of groups of the second class and not sufficiently peculiar to warrant listing them separately. On the other hand the six groups $G_2(AlK)$, $G_3(K)$, $G_3(AlS)$, $G_3(Al)_{r=-1}$, $G_5(A)_{r=-1}$ and $G_5(l)_{r=-1}$ are essentially distinct from the groups listed in the second class.

THEOREM 56. There are six distinct varieties of groups of tpye I, third class, viz.: $G_2(AlK)$, $G_3(K)$, $G_3(AlS)$, $G_3(Al)_{r=-1}$, $G_5(A)_{r=-1}$ and $G_5(l)_{r=-1}$.

B. GROUPS OF TYPES II AND III.

We pass now to the problem of determining a complete list of the varieties of groups of type II. We have already found in § 5 a complete list of the different varieties of groups of type II defined by sets of linear relations on the elements of M with the additional condition $D=0$. In § 7 we discussed the fundamental group of type II, $G_2'(AA'l)$, and its subgroups $G_1'(AA'l)_a$ and found, Art. 211, that no collineation of type II can leave a conic invariant; hence there are no groups of type II defined by quadratic relations on the elements of M.

299. *Two Classes of Groups of Type II.* We shall find two distinct classes of groups of type II, viz.: (1) those in which the two parameters k and t are independent of each other; and (2) groups made up of collineations in each of which the

parameters k and t satisfy the relation $k = a^t$ for a constant value of a. Groups of the first class can all be built up out of the ∞^5 two-parameter group $G_2'(AA'l)$ in the plane; groups of the second class can all be built up out of one-parameter groups $G_2'(AA'l)_a$.

300. *Groups of the First Class.* Our list of groups of the first class is already known; it consists of the six groups named in Theorem 22, viz.: $G_4'(Al')$, $G_4(Al)$, $G_4'(A'l')$, $G_8'(AA')$, $G_3'(ll')$, $G_2'(AA'l)$. We wish, however, to verify the correctness of this list by means of the normal form of type II. Since each of these groups is reducible, this is an easy task. In this way we shall also find the k and t-relations that enable us to determine the list of groups of the second class.

We proved in Theorem 21 that a necessary and sufficient condition for a group of type II is that all collineations in the system shall have in common the same invariant lineal element; and we drew the conclusion that there are three distinct varieties of groups of type II with invariant lineal element, one for each lineal element in the figure $(AA'l)$. We shall verify this conclusion by means of the normal form of type II.

301. *The Group $G_4'(Al)$.* Let T' and T_1' be so chosen that their invariant figures have the point A and the line l in common; and let A be the origin and l the y-axis. The normal form of T' reduces to

$$x_1 = \frac{kx}{1 + ty + \left(\frac{k-1}{A'} - \frac{B'}{A'}t\right)x} \text{ and } y_1 = \frac{y + \frac{B'}{A'}(k-1)x}{1 + ty + \left(\frac{k-1}{A'} - \frac{B'}{A'}t\right)x} \quad (76)$$

Writing T_1' in the same form as T' and eliminating x_1 and y_1, the resultant is found to be of the same form with the following conditional equations:

(1) $k_2 = kk_1$. (2) $t_2 = t + t_1$.

$$(3)\ \frac{B_2'}{A_2'}(k_2 - 1) = \frac{B'}{A'}(k-1) + k\frac{B_1'}{A_1'}(k_1 - 1). \quad (77)$$

(4) $\frac{k_2-1}{A_2'} - \frac{B_2'}{A_2'}t_2 = \frac{k-1}{A'} + \frac{k(k_1-1)}{A_1'} - \frac{B'}{A'}(t+t_1) - kt\left(\frac{B'}{A'} + \frac{B_1'}{A_1'}\right).$

Hence the existence of the group $G_4'(Al)$ is verified.

302. *The Group* $G_4(A'l')$. Let T' and T_1' be so chosen that their invariant figures have the point A' and the line l' in common; and let A' be the origin and l' the x-axis. The normal form of T' reduces to

$$x_1 = \frac{x + \frac{At}{c'}y}{k + \frac{1-k}{A}x + \left(\frac{t}{c'} + \frac{c}{c'}\frac{k-1}{A}\right)y},$$

$$y_1 = \frac{y}{k + \frac{1-k}{A}x + \left(\frac{t}{c'} + \frac{c}{c'}\frac{k-1}{A}\right)y}. \qquad (78)$$

A simplification of the normal form results if we set $\frac{At}{c'} = t'$. It was proved in chapter II, Art. 136, that, if t and t' denote the parabolic constants along l' and through A' respectively, $t' = \frac{AA'}{\sin\phi}t$ where ϕ is the angle lAA'. In the above normal form A is the distance between the invariant points and c' the sine of the angle between the invariant lines; hence t' in the equation $t' = \frac{At}{c'}$ is the parabolic constant through A'. Making this substitution equations (78) reduce to

$$x_1 = \frac{x + t'y}{k + \frac{1-k}{A}x + \left(\frac{t'}{A} + \frac{c}{c'}\left(\frac{k-1}{A}\right)\right)y} \qquad y_1 = \frac{y}{(\textit{Same denominator.})} \qquad (78')$$

Writing T_1' in the same form and eliminating, the resultant is also in the same form with the following conditional equations:

(1) $k_2 = kk_1$. (2) $t_2' = t' + t_1'$.

(2) $\frac{1-k_2}{A_2} = \frac{1-k_1}{A_1} + \frac{k_1(1-k)}{A}$. (79)

(4) $\frac{t_2'}{A_2} + \frac{c_2}{c_2'}\frac{k_2-1}{A_2} = k_1t'\left(\frac{1}{A} - \frac{1}{A_1}\right) +$

$$\frac{1}{A_1}(t'+t_1') + \frac{c_1}{c_1'}\left(\frac{k_1-1}{A_1}\right) + \frac{c}{c'}k_1\left(\frac{k-1}{A}\right).$$

Hence the existence of the group $G_4'(A'l)$ is verified.

303. *The Group* $G_4'(A'l)$. Let T' and T_1' be so chosen that their invariant figures have the point A and the line l' in common, and let A be the origin and l' the x-axis. The normal form of T' reduces to

$$x_1 = \frac{kx + \frac{c}{c'}(1-k)\,y}{1 + \frac{k-1}{A'}\,x + \left\{\frac{t}{c'} + \frac{c}{c'}\,\frac{(1-k)}{A'}\right\} y} \qquad y_1 = \frac{y}{(\textit{Same denominator.})} \tag{80}$$

Writing T_1' in the same form and eliminating, the resultant is also of the same form with the following conditional equations:

$$\begin{aligned}
&(1)\ k_2 = kk_1.\\
&(2)\ \frac{k_2-1}{A_2'} = \frac{k-1}{A'} + k\,\frac{k_1-1}{A_1'}.\\
&(3)\ \frac{c_2}{c_2'}(1-k_2) = \frac{c}{c'}k_1(1-k) + \frac{c'}{c_1'}(1-k_1').\\
&(4)\ \frac{t_2}{c_2'} + \frac{c_2}{c_2'}\,\frac{(1-k_2)}{A_2'} = \frac{t}{c'} + \frac{t_1}{c_1'} + \frac{c}{c'}\,\frac{(1-k)}{A'}\\
&\qquad\qquad + \frac{c}{c'}(1-k)\,\frac{(k_1-1)}{A_1'} + \frac{c_1}{c_1'}\,\frac{(1-k_1)}{A_1'}.
\end{aligned} \tag{81}$$

Hence the existence of the group $G_4(Al')$ is verified.

304. *The Group* $G_3'(AA')$. If we make $A_1 = A$ in the conditional equations of the group $G_4'(A'l')$ we find the conditional equations of the group $G_3'(AA')$ as follows:

$$\begin{aligned}
&(1)\quad k_2 = kk_1, \qquad\qquad (2)\quad t_2' = t' + t_1,\\
&(3)\quad \frac{c_2}{c_2'}(k_2-1) = \frac{c}{c'}k_1(k-1) + \frac{c_1}{c_1'}(k_1-1).
\end{aligned} \tag{82}$$

We also obtain the same equation by making $A_1' = A'$ in the conditional equations of the group $G_4'(Al)$ and writing t for $\frac{At}{c'}$. Hence the existence of the group $G_3'(AA')$ is proved, and also that it is a subgroup of both $G_4'(A'l')$ and $G_4'(Al)$.

305. *The Group* $G_3'(ll')$. If we make $B' = 0$ in the normal form of the group $G_4'(Al)$, we obtain the normal form of a group $G_3'(ll')$ as follows:

$$x_1 = \frac{kx}{1 - ty + \frac{k-1}{A'}x}, \qquad y_1 = \frac{y}{1 + ty + \frac{k-1}{A'}x}. \tag{86}$$

Making $B' = 0$ and $B_1) = 0$ in the conditional equations of $G_4(Al)$ we get

$$(1)\quad k_2 = kk_1, \quad (2)\quad t_2 = t + t_1,$$
$$(3)\quad \frac{k_2-1}{A_2'} = \frac{k-1}{A'} + \frac{k(k_1-1)}{A_1'}. \tag{84}$$

The same normal form and conditional equation may be obtained by making $c = 0$, $c' = 1$, $c_1 = 0$ and $c_1' = 1$, in the normal form of the group $G_4'(Al')$ and its conditional equations.

306. *The Group* $G_2'(AA'l)$. If we make $A_1' = A'$ in the normal form and conditional equations of the group $G_3'(ll')$, we fix the point A' on the x-axis and obtain the normal form and conditional equations of $G_2'(AA'l')$, the fundamental group of type II. The normal form remains the same as in $G_3'(ll)$ and the conditional equations become

$$(1)\quad k_2 = kk_1, \qquad (2)\quad t_2' = t + t_1. \tag{85}$$

The same results may be obtained by making $c = 0$, $c' = 0$, $c_1 = 1$ and $c_1' = 1$, in the normal form and conditional equations of $G_3'(AA')$.

We have now verified the list of all varieties of groups of type II that can be compounded out of the ∞^3 two-parameter groups $G_2'(AA'l')$.

THEOREM 57. There are six varieties of groups of the first class of type II, viz.: $G_2'(AA'l)$, $G_3'(AA')$, $G_3'(ll)$, $G_4'(Al)$, $G_4'(A'l')$ and $G_4'(Al')$.

307. *Groups of the Second Class.* It was shown in Art. 258, that the group $G_2'(AA'l')$ breaks up into one-parameter groups $G_1'(AA'l')_a$ when we put $k = a^t$ and keep a constant. In like manner all groups of the first class of type II, which have among their conditional equations these two, viz.: $k_2 = kk_1$ and $t_2 = t + t_1$, break up into subgroups characterized by a constant a. The groups of the first class which meet these conditions are $G_2'(AA'l')$, $G_2'(AA))$, $G_3'(ll')$, $G_4'(A'l)$, $G_4'(A'l')$. We therefore have five varieties of groups of the second class, viz.: $G_1'(AA'l)_a$, $G_2'(AA')_a$, $G_2'(ll')_a$, $G_3'(Al)_a$

and $G_3'(A'l')_a$. The group $G_4'(Al')$ does not break up into subgroups of the second class.

THEOREM 58. There are five varieties of groups of the second class of type II, viz.: $G_1'(AA'l')_a$, $G_2'(AA')_a$, $G_2'(ll')_a$, $G_3'(Al)_a$ and $G_3'(A'l')_a$.

308. *Groups of Type III.* We found in Art. 227, that there is only one variety of group of type III defined by linear relations on the elements of M and $D = 0$ and $D' = 0$. This is the group $G_3''(Al)$ which is the fundamental group of the type. This fundamental group was investigated in § 7, and it was found to contain two varieties of subgroups, viz.: $G_2''(AlN)$ and $G_2''(AlS)$. Hence these three varieties of groups of type III form the complete list.

THEOREM 59. There are three varieties of groups of type III, viz.: $G_3''(Al)$, $G_2''(AlN)$, $G_3''(AlS)$.

C. TABLE OF GROUPS OF COLLINEATIONS OF THE PLANE.

In this table, the collineation groups of the plane are classified according to the five types of collineations. Each group is designated by an appropriate symbol. The self-dualistic groups are enclosed in brackets, thus: $\{G(AA'A'')\}$; a pair of dualistic groups are bracketed together, thus: $\left\{\begin{matrix}G_6(A)\\G(l)\end{matrix}\right\}$. Similar tables are given by Lie, "Continuierliche Gruppen," pp. 288 291, and by Meyer, "Chicago Congress Papers," pp. 188 190; but in these tables of Lie and Meyer the notation and classification is entirely different. The numbers on the right refer to Lie's and Meyer's tables respectively.

TYPE I.

A.—Groups of the First Class.

	Symbol.	Invariant Figure.	Lie.	Meyer.
(1).	$\{G_2(AA'A'')\}$.	Triangle	(31)	(23)
(2).	$\{G_3(AA'l')\}$.	Two points, their joining line, and a line through one of these points	(20)	(15)
(3).	$\{G_4(AA')\}$.	Two points and their joining line	(11)	(9*a*)
(4).	$\{G_4(ll')\}$.	Two lines and their point of intersection	(12)	(9*b*)
(5).	$\{G_4(A,l')\}$.	Point and line, separate	(10)	(8)
(6).	$\{G_5(Al)\}$.	Lineal element	(6)	(4)
(7).	$\{G_6(A)\}$.	Point	(3)	(2*b*)
(8).	$\{G_6(l)\}$.	Line	(2)	(2*a*)
(9).	$\{G_8\}$.	No invariant figure	(1)	(1)

B.—Groups of the Second Class.

(1).	$\{G_1(AA'A'')_r\}$.	Triangle and pencil of path-curves	(36)	(26)
(2).	$\{G_2(AA'l')_r\}$.	Same as $G_3(AA'l')$	(27)	(20)
(3).	$\{G_3(AA')_r\}$.	Same as $G_4(AA')$	(18)	(14*a*)
(4).	$\{G_3(ll')_r\}$.	Same as $G_4(ll')$	(19)	(14*b*)
(5).	$\{G_4(Al)_r\}$.	Same as $G_5(Al)$	(7)	(5)

C.—Groups of the Third Class.

(1).	$\{G_1(AA'A''K)\}$.	Triangle and pencil of conics having double contact	(35)	(26)
(2).	$\{G_2(AlK)\}$.	Conic and point A on it and tangent at A,	(34)	(25)
(3).	$\{G_3(K)\}$.	Conic	(23)	(17)
(4).	$\{G_3(AlS)\}$.	Lineal element Al and pencil of conics having third order contact at A and touching l	(14)	(11)
(5).	$\{G_3(A,l')_{r=2}\}$.	Same as $G_4(A,l)$	(17)	(13)
(6).	$\{G_4(Al)_{r=2}\}$.	Lineal element	(8)	(6)
(7).	$\{G_5(A)_{r=2}\}$.	Point	(5)	(3*b*)
(8).	$\{G_5(l)_{r=2}\}$.	Line	(4)	(3*a*)

TYPE II.

A.—Groups of the First Class.

(1).	$G_2'(AA'l)$.	Same as $G_3(AA'l')$	(28)	(21)
(2).	$G_3'(AA')$.	Same as $G_4(AA')$	(18)	(14*a*)
(3).	$G_3'(ll')$.	Same as $G_4(ll')$	(19)	(14*b*)
(4).	$G_4'(Al)$.	Lineal element	(7)	(5*a*)
(5).	$G_4'(A'l)$.	" "	(7)	(5*b*)
(6).	$G_4'(Al)$.	" "	(9)	(7)

B.—Groups of the Second Class.

(1).	$G_1'(AA'l)_a$.	Same as $G_3(AA'l)$ and pencil of path-curves	(36)	(27)
(2).	$G_2'(AA')_a$.	Same as $G_4(AA')$	(25)	(19*a*)
(3).	$G_2'(ll')_a$.	Same as $G_4(ll')$	(26)	(19*b*)
(4).	$G_3'(Al)_a$.	Lineal element	(15)	(12*a*)
(5).	$G_3'(A'l)_a$.	" "	(16)	(12*b*)

TYPE III.

(1).	$G_1''(AlS)$.	Lineal element Al and pencil S of ∞^1 conics having contact of third order with l at A	(37)	(28)
(2).	$G_2''(AlN)$.	Lineal element Al and net N of ∞^2 conics having contact of second order with l at A	(24)	(18)
(3).	$G_3''(Al)$.	Lineal element Al and ∞^3 conics touching l at A	(13)	(10)

TYPE IV.

(1).	$H_1(Al)$.	All points on l and all lines through A	(38)	(29)
(2).	$H_2(ll')$.	The lines l and l' and all points on l	(32)	(24*a*)
(3).	$H_2(AA')$.	The points A and A' and all lines through A,	(33)	(24*b*)
(4).	$H_3(A)$.	All lines through A	(22)	(16*b*)
(5).	$H_3(l)$.	All points on l	(21)	(16*a*)

TYPE V.

(1).	$H_1'(Al)$.	All points on l and all lines through A	(39)	(30)
(2).	$H_2'(A)$.	All lines through A	(30)	(22*b*)
(3).	$H_2'(l)$.	All points on l.	(29)	(22*a*)

A comparison of the tables of Lie and Meyer show them to be practically identical; in fact, Meyer has only put Lie's infinitesimal notation into finite form. A comparison of the present table with Lie's table shows some results worthy of notice.

The groups numbered (18) and (19) in Lie's table are the groups $G_3(AA')_r$ and $G_3(ll')_r$ for all values of r. These are, in general, of type I, second class, but they include also the groups $G_3'(AA')$ and $G_3'(ll')$ of type II, first class. The existence of these latter groups would hardly be suspected from Lie's or Meyer's formulæ.

Again, Lie's group (7) is in general, $G_4(Al)_r$ for all values of r, but it also includes the groups $G_4'(Al)$ and $G_4'(Al')$. The existence of the latter as distinct groups is unknown to Lie's theory.

The group $G_4(Al)_{r=2}$, (8) of Lie's table, is only a special case of $G_4(Al)_r$, (7) of the same table; and, though its structure is somewhat peculiar, it is doubtful if it is worthy of special mention in the list.

§ 10. Groups of Real Collineations.

Thus far in treating collineations in a plane we have considered the most general case where variables and parameters are complex numbers. We shall now examine the special case of real collineations, *i. e.*, those that transform real points into real points.

309. *The Real Group G_8.* A real collineation is represented analytically by the equations

$$x_1 = \frac{a_1 x + b_1 y + c_1}{a_3 x + b_3 y + c_3}, \qquad y_1 = \frac{a_2 x + b_2 y + c_2}{a_3 x + b_3 y + c_3}, \tag{1}$$

where variables and coefficients are all real numbers. If the coefficients are all real, the point (x_1, y_1) will be real when, and only when, (x, y) is a real point. The resultant of any two real collineations is also real and hence all real collineations in a plane form a real group RG_8. This real group is a subgroup of G_8, the general projective group of the plane.

A list of the subgroups of RG_8 will be found to be nearly identical with the list of subgroups of G_8 given in § 6.

THEOREM 60. All real collineations in a plane form a group RG_8.

310. *Hyperbolic and Elliptic Collineations of Type I.* A collineation of type I leaves a triangle invariant, as was shown in chapter II, Arts. 98, 103. The coordinates of the vertices of the invariant triangle were found by solving the cubic equation

$$\alpha x^3 + \beta x^2 + \gamma x + \delta = 0.$$

In the case of a real collineation the coefficients of this equation are all real, since they are rational functions of the real coefficients of the collineation. The roots of this equation, when unequal, may be all real or one real and two conjugate imaginary; the same is also true of the equation giving the y-coordinates of the invariant points. Hence there are two varieties of real collineations of type I; one whose invariant triangle is real in all of its parts, and the other whose invariant triangle has one real and two conjugate imaginary vertices, one real and two conjugate imaginary sides. We shall call these *Hyperbolic* and *Elliptic* collineations respectively.

THEOREM 6. A real collineation of type I is either hyperbolic or elliptic.

311. *The Hyperbolic Group $hG_2(AA'A'')$.* All collineations leaving a real triangle invariant form a group $hG_2(AA'A'')$. The one-dimensional transformations along the three sides are all hyperbolic, each with two real invariant points and a real cross-ratio k (Chap. I, Art. 39). Hence the hyperbolic collineation hT has a real invariant triangle and real cross-ratio parameters k and k'.

Since the parameters k and k' assume all real values, there are ∞^2 collineations in the group $hG_2(AA'A'')$. By setting $k' = k^r$ we see that $hG_2(AA'A'')$ contains ∞^1 one-parameter subgroups $hG_1(AA'A'')_r$; since k and k' are both real, r must also be real. The constant r may assume the six critical values $(1, 0, \infty)$ and $(-1, 2, \frac{1}{2})$ which give the three perspective subgroups of $hG_2(AA'A'')$, and the three subgroups whose path-curves are conics.

312. *The Elliptic Group* $eG_2(AA'A'')$. Let A be a real point and A' and A'' a pair of conjugate imaginary points; then l'' joining A' and A'' is a real line and l and l' joining AA' and AA'' respectively are a pair of conjugate imaginary lines. The cross-ratios k and k' may be shown to be conjugate imaginary numbers. The implicit normal form of T was given in Chapter II, Art. 129, by the equations

$$\frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A'' & B'' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}}; \quad \frac{\begin{vmatrix} x_1 & y_1 & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x_1 & y_1 & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} = k' \frac{\begin{vmatrix} x & y & 1 \\ A & B & 1 \\ A' & B' & 1 \end{vmatrix}}{\begin{vmatrix} x & y & 1 \\ A' & B' & 1 \\ A'' & B'' & 1 \end{vmatrix}} \qquad (86)$$

Since A' and A'', (B' and B'') differ only in the sign of i, being conjugate imaginaries, k and k' can differ only in the sign of i and are also conjugate imaginaries. Hence k' is not independent of k; but k may be written in the form of $\rho e^{i\theta}$ and depends therefore upon two independent quantities, ρ and θ; and hence there are ∞^2 elliptic collineations leaving the triangle $(AA'A'')$ invariant. These form the elliptic group $eG_2(AA'A'')$. There is one collineation in $eG_2(AA'A'')$ corresponding to each value of the complex number k, or, speaking geometrically, to each point in the complex plane.

To show how $eG_2(AA'A'')$ breaks up into one-parameter subgroups we proceed as follows: Let $k = \rho e^{i\theta} = e^{(c+i)\theta}$; since $k_2 = kk_1$, we have

$$e^{(c_2+i)\theta_2} = e^{c\theta + c_1\theta_1 + i(\theta+\theta_1)}.$$

If T_1 is chosen so that $c_1 = c$, then c_2 also equals c and $\theta_2 = \theta + \theta_1$; *i. e.*, $e^{(c+i)\theta} = e^{(c+i)(\theta+\theta_1)}$. Hence if c be kept constant and θ alone varies, we have a one-parameter subgroup $eG_1(AA'A'')_c$. The two-parameter group $eG_2(AA'A'')$ contains ∞^1 one-parameter subgroups, one for each value of c in $k = e^{(c+i)\theta}$.

There are two subgroups of $eG_2(AA'A'')$ of special importance, viz.: for $c = o$ and $c = \infty$. When $c = o$, $k = e^{i\theta}$ and $k' = e^{-i\theta}$; in order that the equation $k' = k^r$ should be satisfied by these values of k and k', we must put $r = -1$. Hence the path-curves of the group $eG_1(AA'A'')_{c=o}$ are a pencil of conics having double contact at a pair of conjugate imaginary points A' and A''. The conics of such a pencil have no real points in common. The conics of such a pencil are either real or pure imaginary without a real point.

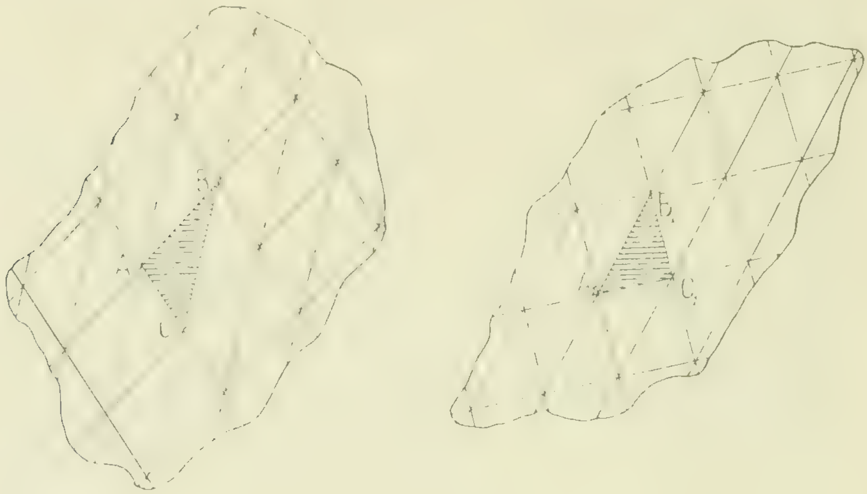

FIG. 31.

If c approach ∞ and θ approach o at the same time, $c\theta$ may approach a finite number n. In such a case $k = e^n$, a real number.

The cross-ratio k'' along the line $A'A''$ must be of the form e^{i}, since the one-dimensional transformation along that side is elliptic. Since $k' = kk''$, $k'' = \frac{k'}{k} = \frac{\rho e^{-i\theta}}{\rho e^{i\theta}} = e^{-2i\theta}$. When $\theta = o$, $k'' = 1$ and the transformation along $A'A''$ is identical. Hence the subgroup of $eG_2(AA'A'')_{c=\infty}$ is a group $H_1(Al'')$ of real perspective collineations.

THEOREM 62. The real elliptic group $eG_2(AA'A'')$ contains one subgroup whose path-curves are conics and one whose path-curves are straight lines.

313. *Real Groups of Type I.* Having determined two varieties of fundamental groups of type I we go on to enumerate the groups of higher order that can be compounded out of these.

The invariant figures (A), (l), (Al), (AA'), (ll'), (A,l''), $(AA'l')$ must be examined separately. The real groups $G_6(A)$, $G_6(l)$ and $G_7(A,l'')$ contain both hyperbolic and elliptic collineations. The groups $G_5(Al)$ and $G_3(AA'l')$ contain hyperbolic, but no elliptic, collineations of type I. There are two varieties of groups leaving two points invariant, viz.: $hG_4(AA')$ and $eG_4(A'A'')$. In the first case the two points A and A' are real and the group $hG_4(AA')$ contains only hyperbolic collineations of type I (and lower types). In the second case the points A' and A'' are conjugate imaginary points and $eG_4(A'A'')$ contains only elliptic collineations of type I. In like manner we have two varieties of groups leaving a pair of lines invariant, viz.: $hG_4(ll')$ and $eG_4(ll')$.

Evidently the groups $hG_4(AA')$ and $hG_4(ll')$ break up into subgroups of the second class for a constant r, just as $hG_2(AA'A'')$ does. Also the groups $eG_4(A'A'')$ and $eG_4(ll')$ break up into subgroups of the second class for a constant c just as $eG_2(AA'A'')$ does.

The real groups of the third class $G_5(A)_{r=-1}$, $G_5(l)_{r=-1}$ and $G_3(A,l'')_{r=-1}$, contain both hyperbolic and elliptic collineations of type I. The real groups $G_2(AlK)$ and $G_3(AlS)$ contain only hyperbolic collineations of type I. There are two varieties of real three-parameter groups leaving a conic K invariant, viz.: $h\,G_3(K)$ and $e\,G_3(K)$. In the first case the conic K is real and $e\,G\,(K)$ contains only hyperbolic collineations of type I. In the second case the conic is imaginary, having no real points, and $e\,G_3(K)$ contains only elliptic collineations of type I.

THEOREM 63. The real groups $G_4(AA')$, $G_4(ll')$, $G_3(K)$, and $G_2(AA'A'')$ exist in two varieties each, viz.: hyperbolic and elliptic.

314. *Real Collineations of Types II, III, IV and V.* A real collineation of type II leaves invariant a figure $(AA'l)$ real in all of its parts. There is, therefore, only one variety of collineations of this type. Type II appears as the parabolic case between the hyperbolic and elliptic cases of type I.

A real collineation of type III leaves invariant a real lineal element Al and a real pencil of conics S. This type appears as the parabolic case between the hyperbolic and elliptic cases of type I when the path-curves are conics.

A real perspective collineation of type IV or V leaves invariant a real axis, vertex and pencil of lines.

A list of the real groups of types II, III, IV and V, is identical with the list given on page 254, of the present chapter.

Exercises on Chapter III.

1. Give several examples of systems of collineations which possess the first group property but not the second.

2. Give examples of systems of collineations that possess the second group property but not the first.

3. Discuss in detail the properties of $H_1(A,l)$, the fundamental group of type IV.

4. Discuss in detail the properties of $H_1'(Al)$, the fundamental group of type V.

5. Show that through a given point P of the plane (not a vertex of the triangle $AA'A''$) there passes one and only one path-curve of the group $G_1(AA'A'')_r$.

6. Show that one and only one path-curve of the group $G_1(AA'A'')_r$ touches a given line l (not a side of the triangle $AA'A''$) of the plane.

7. Show that the constant r in the one-parameter group $G_1(AA'A'')_r$ is the cross-ratio of the four points $(TLL'L'')$ on any line g of the plane, where T is the point of the path-curve touching g and L, L', L'', are the points where g cuts the sides of the invariant triangle $(AA'A'')$.

8. Show that an involutoric collineation in its plane is necessarily of type IV.

9. From the self-dualistic character of a collineation show that, when the path-curves consist of a pencil of conics, they must have double contact with each other and with the invariant triangle.

10. What is the geometrical meaning of a in the equation $x = Cya^{\frac{1}{A}\frac{x}{y}}$?

11. Develop the whole theory of perspective collineations by the methods suggested in Article 276.

12. Show that $2a$ is the common radius of curvature at the origin of all conics of the system

$$x^2 + 2hxy + 4aCy^2 = 4ay.$$

13. Show that each collineation of type III in $G_3''(Al)$ can be resolved into two elations, one belonging to the group $H_2'(A)$ and the other to $H_2'(l)$.

14. Show that the group $G_2''(AlN)$ contains as a subgroup the group of elations $H_1'(Al)$.

15. Verify equations (24) of Article 217; prove that the determinant of a complete family of automorphic forms of degree n is equal to Δ^{n^2}.

16. Verify equation (35^a_9) of Article 217 and factor the general determinant $\Delta_n(1) = 0$.

17. Deduce the equations and give an analytic proof of the existence of the groups $H_2(ll')$ and $H_2(AA')$.

18. Show that equations (36″) represent a collineation of type II.

19. Show that every collineation of type I belongs to one and only one two-parameter group $G_2(AA'A'')$.

20. Show that two groups $G_2(AA'A'')$ and $G_2(AA_1'A_1'')$, where A', A'', A_1', A_1'' are collinear, have a common subgroup $H_1(Al'')$ of type IV.

21. Show that $G_6(A)$ and $G_6(A'')$ have in common $G_4(AA'')$; $G_6(l)$ and $G_6(l')$ have in common $G_4(ll')$; $G_6(A)$ and $G_6(l'')$ have in common $G_4(A, l'')$.

22. Show that the following groups bracketed together are dualistic:

$$\left\{\begin{matrix} G(AA'l)_r \\ G(AA'l)_r \end{matrix}\right. ,\quad \left\{\begin{matrix} G(AA')_r \\ G(ll')_r \end{matrix}\right. ,\quad \left\{\begin{matrix} G(Al)_r \\ G(Al)_r \end{matrix}\right. .$$

23. Show that the following groups are self-dualistic: $G(AA'l)_r$, $G(Al)$.

24. Show that the two collineations of type I which are the resultants of T and T' in different orders have the same invariant cross-ratio, but not the same invariant triangle; show that their invariant triangles have equal areas.

25. Show that $G_3''(Al)$ is a subgroup of $G_4(Al)_{r=2}$.

26. Show that $G_1''(AlS)$ is a subgroup of $G_3(AlS)$.

27. Show that $h\,G_2(AA'A'')$ contains three and $e\,G_2(AA'A'')$ only one involutoric collineation.

28. Show that the family of path-curves of the group $e\,G_1(AA'A'')_{r=1}$ contains both real and pure imaginary conics.

29. Either all three cross-ratios along the sides of the invariant triangle of $h\,G_2(AA'A'')$ are positive or one is positive and two negative.

30. The group $G_2(AlK)$ contains no elliptic collineations of type I.

31. The group $e\,G_3(iK)$ contains only elliptic collineations of type I.

CHAPTER IV.

GROUP STRUCTURE AND SOME SPECIAL GROUPS.

§ 1. Structure of the Collineation Groups of the Plane.
§ 2. Singular Transformations.
§ 3. Mixed Groups.
§ 4. Generation of Finite from Infinitesimal Collineations.
§ 5. The General Linear Group $G_6(l\infty)$.
§ 6. The Group $G_3(K)$.
Exercises.

In the last chapter we determined all varieties of collineation groups in the plane and classified them with respect to the five types. In the present chapter we shall examine into the structure of each variety of continuous groups of plane collineations and discuss the generation of such groups from infinitesimal collineations. We shall also discuss in detail two specially important groups, viz.: $G_6(l\infty)$ and $G_3(K)$ and their subgroups.

The structure of each variety of collineation group will be discussed in § 1, and the existence of the so-called singular transformations and their properties will be brought out in § 2. Mixed groups of plane collineations, *i. e.*, groups not continuous but containing continuous subgroups, are treated in § 3. § 4 is devoted to the important topic of the generation of continuous groups of collineations from infinitesimal collineations. The group $G_6(l\infty)$, which leaves the line at infinity invariant, will be discussed in detail in § 5, and the group $G_3(K)$ in § 6.

§1. Structure of the Collineation Groups of the Plane.

Having found a complete list of the groups of collineations in the continuous plane, we must now examine more closely into the structure of each group. Many of these groups will be found to contain collineations of one type only, while others will be found to contain collineations of two or more types. In every case, there are collineations of a characteristic type which make up all or the greater part of the group, and among these are to be found in many cases a smaller number of collineations of one or more lower types. In most instances, the collineations of these secondary types form continuous subgroups of the given group; but sometimes these secondary collineations in a given group do not form a continuous subgroup, in which case they are called *Singular Transformations.*

The complete structure of some of these groups has already been given, while in other cases only the characteristic collineations of the group have been indicated. The entire list of these groups should be examined; usually the structure of a group will be given without proof, but in a few typical cases where the structure is not at once evident, or where the group contains singular transformations, the proofs will be indicated. The verification of the structural formulas in the remaining groups will be left as exercises for the reader.

315. *Structure of the Perspective Groups.* We found in Chapter III, Art. 284, eight varieties of perspective groups, viz.: three of type V and five of type IV. The group $H_1'(Al)$ contains only collineations of type V, and $H_1(A, l'')$ only those of type IV. The structure of the other six perspective groups is here indicated by symbolic equations

$$\begin{aligned} H_2'(l) &= \infty^1 H_1'(\overline{A}l), \\ H_1'(A) &= \infty^1 H_1'(A\overline{l}), \\ H_2(ll') &= \infty^1 H_1(\overline{A}, l) + H_1'(A'l), \end{aligned}$$

$$
\begin{aligned}
H_2(AA') &= \infty^1 H_1(A,\overline{l}) + H_1'(Al'),\\
H_2(l) &= \infty^2 H_1(\overline{A},l) + H_2'(l),\\
H_3(A) &= \infty^2 H_1(A,\overline{l}) + H_2'(A).
\end{aligned}
$$

The correctness of these structural formulæ may be proved in detail; we shall give two methods of proof, applying both methods to the same case.

(1) *Synthetic Method.* Take for example the group $H_2(ll')$; we will show that this group must contain collineations of type V. Take from the group $H_2(ll')$ two collineations of type IV, $S(A, l)$ and $S(A_1, l)$ (where A and A_1 lie on l, and l and l' intersect in A'), for which the cross-ratios along l are k and $1/k$ respectively. Along l we have two one-dimensional loxodromic transformations having one and only one invariant point A' in common, hence, Chapter I, Article 32, their resultant is a parabolic transformation, leaving A' invariant, whose parabolic constant t has the value

$$t = (1-k)\left(\frac{1}{A'A_1} - \frac{1}{AA'}\right). \tag{1}$$

Since both transformations along l are identical, their resultant is also identical. Hence, the resultant of S and S_1 is an elation $S(A'l)$, which belongs to the group $H_1'(Al)$. From the above value of t we see that by varying the value of k, or the position of the points A and A_1, all elations in the group $H_1'(Al)$ are obtained; thus, the group $H_2(lA'l)$ contains $H_1(Al)$ as a subgroup. In a similar manner the structural formulæ of the other perspective groups may be verified.

(2) *Analytic Method.* If A' be taken as the origin and l and l' as axes of x and y respectively, the normal forms of $S(Al)$, $S_1(A_1, l)$ and their resultant are respectively

$$S:\ x = \frac{x}{1+\frac{k-1}{B}y}, \qquad y_1 = \frac{ky}{1+\frac{k-1}{B}y}; \tag{2}$$

$$S_1:\ x_2 = \frac{x_1}{1+\frac{k_1-1}{B}y_1}, \qquad y_2 = \frac{k_1 y_1}{1+\frac{k_1-1}{B}y_1}; \tag{2'}$$

$$S_2 : x_2 = \frac{x}{1 + \left(\frac{k-1}{B} + \frac{k(k-1)}{B}\right)y}, \quad y_2 = \frac{k k_1 y}{1 + \left(\frac{k-1}{B} + \frac{k(k-1)}{B_1}\right)y}; \quad (2'')$$

whence $k_2 = kk_1$ and $\frac{k_2-1}{B_2} = \frac{k-1}{B} + \frac{k(k_1-1)}{B_1}$. Now let $k_1 = \frac{1}{k}$ in these equations; we have $k_2 = 1$ and $\frac{1-1}{B_2} = (k-1)\left(\frac{1}{B} - \frac{1}{B_1}\right)$; hence $B_2 = 0$. Putting $\lim \frac{k_2-1}{B_2} = t$ we find $t = (1-k)\left(\frac{1}{B_1} - \frac{1}{B}\right)$. Hence S_2 reduces to

$$x_2 = \frac{x}{1+ty}, \quad y_2 = \frac{y}{1+ty}; \quad (3)$$

and is therefore an elation. Since t may be made to assume all values by varying K or B and B_1, all elations of the group $H_1'(A'l)$ are contained in $H_2(ll')$. The same method may be used to verify the structural formulæ of the other perspective groups.

316. *Structure of Groups of Type III.* We found only three varieties of groups of type III, viz.: $G_1''(AlS)$, $G_2''(AlN)$, $G_3''(Al)$; these cases are easily disposed of. The group $G_1''(AlS)$ contains only collineations of type III. The group $G_2''(AlN)$ contains the group $H_1'(Al)$ as a subgroup (see exercise 14 at the end of the preceding chapter). The group $G_3''(Al)$ contains, as we saw in Article 268, the groups $H_2'(A)$ and $H_2'(l)$ as subgroups. The structural formulæ of the groups of type III are shown as follows:

$$G_2''(AlN) = \infty^1 G_1''(AlS) + H_1'(Al).$$
$$G_3''(Al) = \infty^1 G_1''(AlS) + H_1'(A) + H_2'(l).$$

317. *Structure of Groups of Type II. First Class.* There are six varieties of groups of this class; we have found that the group $G_2'(AA'l')$ contains the groups $H_1(A', l')$ and $H_1(Al)$ as subgroups; and hence we must expect that subgroups of types IV and V will appear in the groups of higher orders of this class. The structural formulæ of these groups are as follows:

$$
\begin{aligned}
G_2'(AA'l') &= G_2'(AA'l') + H_1(l') + H_1'(Al),\\
G_3'(AA') &= \infty^1 G_2'(AA'\overline{l'}) + H_2(lA) + H_2(l),\\
G_3'(ll') &= \infty^1 G_2'(A\overline{A'}l') + H_2(ll') + H_2(A),\\
G_4'(Al') &= \infty^2 G_2'(A\overline{A'}l') + H_3(l') + H_2'(A) + H_2'(l') + G_3''(Al'),\\
G_4'(A'l) &= \infty^2 G_2'(\overline{A}A'l') + H_3(A') + H_2'(l) + H_2'(A') + G_3''(A'l),\\
G_4'(Al) &= \infty^2 G_2'(\overline{A}\overline{A'}\overline{l'}) + \infty^2 H_1(\overline{A'},\overline{l'}) + H_2'(A) + H_2'(l) + G_3''(Al).
\end{aligned}
$$

Synthetic Method. As another example, let us examine the structure of the group $G_4'(Al')$. The point A' may take ∞^2 different positions in the plane, and for each position of A' there is a group $G_2'(AA'l')$ which belongs to the group $G_4'(Al')$. The ∞^2 one-parameter perspective groups $H_1(A',l')$, contained in these groups $G_2'(AA'l')$, belong to the group $H_3(l')$. The group $H_3(l')$ contains the subgroup $H_2'(l')$. The ∞^1 groups of elations $H_1'(Al)$, contained in the groups $G_2'(AA'l')$, form the group $H_2'(A)$ which is, therefore, contained in $G_4'(Al')$. Since $G_4'(Al')$ contains $H_2'(A)$ and $H_2'(l)$, it must also contain $G_3''(Al')$. Thus the structure of $G_4'(Al')$ is found. A similar course of reasoning leads to the structural formulæ of the other groups of this class.

Analytic Method. The normal form of a collineation T of type II in $G_4'(Al')$ is, Art. 137,

$$
T' : x_1 = \frac{kx}{1 + ty + \left(\frac{k-1}{A'} - \frac{B'}{A'}t\right)x}, \quad Y_1 = \frac{y + \frac{B'}{A'}(k-1)x}{1 + ty + \left(\frac{k-1}{A'} - \frac{B'}{A'}t\right)x} \tag{4}
$$

If $t = 0$ in equations (4) these become

$$
x_1 = \frac{kx}{1 + \frac{k-1}{A'}x}, \quad y_1 = \frac{y + \frac{B'}{A'}(k-1)x}{1 + \frac{k-1}{A'}x}, \tag{4a}
$$

which are the equations of the group $H_3(l')$.

If $k = 1$ and $A' = 0$ in (4a) while $\lim. \frac{k-1}{A'} = t'$, then these equations reduce to

$$
x_1 = \frac{x}{1 + t'x}, \quad y_1 = \frac{y + B't'x}{1 + t'x}. \tag{4b}
$$

These are the equations of the group $H_2'(l')$.

If $k = 1$ and $A' \neq 0$ in equations (4) these reduce to

$$x_1 = \frac{x}{1 + ty - \frac{B'}{A'}tx}, \quad y_1 = \frac{y}{1 + ty - \frac{B'}{A'}tx}, \tag{4c}$$

which are the equations of the group $H_2'(A)$,

If $k = 1$, $A' = 0$, $B' = 0$, $\lim. \frac{B'}{A'} = n$ and $\lim. \frac{k-1}{A'} = t'$ in equations (4) these reduce to

$$x_1 = \frac{x}{1 + ty + (t' - nt)x}, \quad y_1 = \frac{y + nt'x}{1 + ty + (t' - nt)x}. \tag{4d}$$

These last equations are the equations of the group $G_3''(Al')$ although not expressed in terms of the natural parameters as in Art. 268. The same method is applicable to the other groups of this class.

318. *Second Class.* The group $G_1'(AA'l')a$, where $a \neq 0$ or ∞, contains only collineations of type II. The structure of the other four groups of this class is shown as follows:

$$G_2'(AA')a = \infty^1 G_1'(A\overline{A}'l')a + H_1'(A'l) + \text{S. T.},$$
$$G_3'(ll')a = \infty^1 G_1'(\overline{AA}'l')a + H_1'(Al') + \text{S. T.},$$
$$G_3'(Al')a = \infty^2 G_1'(A\overline{A}'l')a + H_2'(l') + \text{S. T.},$$
$$G_3'(Al')a = \infty^2 G_1'(\overline{A}A'l)a + H_2'(A') + \text{S. T.}$$

In this class of groups we meet, for the first time, with so-called singular transformations S. T.; these will be discussed later.

Synthetic Method. In order to show that the group of elations $H_2'(l')$ is contained in $G_3'(Al')_a$ we choose the collineations $T'(AA'l')_a$ and $T_1'(AA_1'l')_a$ for which the parameters k and t are (a^t, t) and $(a^{-t}, -t)$ respectively, and form the resultant. The resultant of the two one-dimensional parabolic transformation along l' is identical, *i. e.*, $t_2 = t - t = 0$; that through A is parabolic, since the two pencils through A have the invariant line l in common and $k_1 = a^{-t} = \frac{1}{k}$. Hence the resultant of T' and T_1' is an elation $S'(A_1l')$, where A_1 is some point on l'. By varying the values of t and the positions of A' and A_1' all elations in the group $H_2'(l')$ may be produced.

Analytic Method. Let $k = a^t$, $t = 0$ and $A' = 0$ but *lim* $\frac{a^t - 1}{A'} = t'$ in equations (4) $G_1'(Al')$ Article 317; these reduce to

$$x_1 = \frac{x}{1 + t'x} \quad \text{and} \quad y_1 = \frac{y + B't'x}{1 + t'x}. \tag{5}$$

These equations represent the group of elations $H_2'(l)$.

In like manner the structural formulæ for the remaining groups of this class may be verified.

319. *Structure of Groups of Type I. First Class.* The nine groups of type I, first class, show the following structural formulæ:

$$\begin{aligned}
G_2(AA') &= G_2(AA'A'') + H_1(Al'') + H_1(A'l') + H_1(A''l),\\
G_3(AA'l') &= \infty^1 G_2(A) + G_2'(Al') + H_2(ll') + H_2(A) + H_1(l) + H_1'(Al),\\
G_4(A) &= \infty^3 G_2(A) + \infty^1 G_2'(AA'l') + H_3(l) + H_2'(l) + 2H_2(A),\\
G_4(ll') &= \infty^2 G_2(A) + 2\infty^1 G_2'(Al') + H_3(A) + H_2'(A) + 2H_2(ll'),\\
G_4(A, l'') &= \infty^2 G_2(A) + \infty^1 G_2'(Al'') + \infty^1 H_2(A) + H_1(A, l'')\\
&\qquad + \infty^1 H_1'(l),\\
G_5(Al) &= \infty^3 G_2(A) + \infty^2 G_2'(Al') + \infty^1 G_2'(Al') + G_3''(Al) + H_3(l)\\
&\qquad + H_3(A) + H_2'(l) + H_2'(A),\\
G_6(l) &= \infty^4 G_2(A) + \infty^3 G_2'(Al') + \infty^3 G_2'(Al) + \infty^1 G_3''(Al)\\
&\qquad + H_3(l) + \infty^2 H_2(A) + H_2'(l) + \infty^1 H_2'(A),\\
G_6(A) &= \infty^4 G_2(A) + \infty^3 G_2'(Al) + \infty^3 G_1'(Al') + \infty^1 G_3(Al)\\
&\qquad + H_3(A) + \infty^2 H_2(ll') + H_2'(A) + \infty^1 H_2'(l),\\
G_8 &= \infty^6 G_2(A) + \infty^5 G_2'(Al') + \infty^3 G_3''(Al) + \infty^4 H_1(A, l'')\\
&\qquad + \infty^3 H_1'(Al).
\end{aligned}$$

The verification of these structural formulæ presents no special difficulties.

320. *Second Class.* The structural formulæ of the four following groups are exhibited thus:

$$\begin{aligned}
G_2(AA'l')r &= \infty^1 G_1(AA'\overline{A}'')r + H_1'(Al) + \text{S. T.},\\
G_3(AA')r &= \infty^2 G_1(AA'\overline{A}'')r + H_2'(l) + \text{S. T.},\\
G_3(ll')r &= \infty^2 G_1(A\overline{A}'\overline{A}'')r + H_2'(A) + \text{S. T.},\\
G_4(Al)r &= \infty^3 G_1(A\overline{A}'\overline{A}'')r + H_2'(A) + H_2'(l)\\
&\qquad + G_3''(Al) + \text{S. T.}
\end{aligned}$$

Synthetic Method. From the group $G_3(AA')_r$ take two collineations $T(AA'A'')_r$ and $T_1(AA'A_1'')_r$, where A'' and

A_1'' are not collinear with either A or A'. Let $k_1 = \frac{1}{k}$; $k' = k^r$ and $k_1' = k^{-r}$. The resultant of the two one-dimensional transformations along the line AA' is identical since both invariant points are common and the cross-ratios have reciprocal values. The resultants of the one-dimensional transformations of the pencils through both A and A' are parabolic, since they each have only one invariant line in common and reciprocal values of the cross-ratios. Hence the resultant of T and T_1 is an elation $S'(xl)$ where x is some point on l. It is easy to show that all elations in $H_2'(l)$ are contained in $G_3(AA')_r$.

Analytic Method. The equations of the normal form of $G_3(AA')_r$ are

$$x_1 = \frac{kx + \frac{A''}{B''}(k^r - k)\,y}{\frac{k-1}{A'}x + \frac{A''(k^r-1) - A'(k-1)}{A'B''}\,y + 1}, \quad y_1 = \frac{k^r y}{\text{Same denom.}} \tag{6}$$

Let $k = 1$, $B'' = 0$ and $\lim.\frac{k-1}{B''}\left(\frac{k^r-1}{k-1} - \frac{A''}{A'}\right) = t$, then $\lim.\frac{k-1}{B''}\left(\frac{A''k(k^{r-1}-1)}{k-1}\right) = xt$ where x is some constant. Equations (6) reduce to

$$x_1 = \frac{x + aty}{1 + ty}, \quad y_1 = \frac{y}{1 + ty}. \tag{6'}$$

These are the equations of the group $H_2'(l)$.

In like manner the structural formulæ of the other group of this class may be verified.

321. *Third Class.* The list of groups of the third class shows structural formulæ as follows:

$$\begin{aligned}
G_2(AlK) &= \infty^1 G_1(AA'A'') + G_1''(AlS),\\
G_3(K) &= \infty^2 G_1(AA'A'') + \infty^1 G_1''(AlS),\\
G\,(AlS) &= \infty^1 G_1(AA'A'') + G_1''(AlS)\\
&\qquad + H^1(Al) + \text{S. T.},\\
G_3(A, l'')_{r=-1} &= \infty^2 G_1(AA'A'') + \infty^1 H_1'(AAl'') + \text{S. T.},\\
G_5(l'')_{r=-1} &= \infty^4 G_1(AA'A'') + \infty^1 H_2'(Al) + \text{S. T.},\\
G_5(A)_{r=-1} &= \infty^4 G_1(AA'A'') + \infty^1 H_2(l) + \text{S. T.}
\end{aligned}$$

The group $G_3(AlS)$ contains ∞^2 subgroups $G_1(AA'A'')$, where A' is in turn every point on Al and A'' in turn every point in the plane. One conic of the pencil S passes through A'', and $A'A''$ is tangent to this conic at A'. Take a collineation T from the group $G_1(AA'A'')$ and another T_1 from the group $G_1(AA_1'A_1'')$ having cross-ratio constants k and $1/k$ respectively. Their resultant is parabolic along Al and through A, and hence is a collineation T'' of type III belonging to the group $G_1''(Al)$. It can be shown that all the collineations of the group $G_1''(Al)$ are to be found in $G_3(S)$.

Again, take two collineations T and T_1, whose cross-ratio constants are k and $\frac{1}{k}$, from the groups $G_1(AA'A'')$ and $G_1(AA'A_1'')$, where A and A_1 are collinear with A. Their resultant is identical along A and through A, but is parabolic along A_1 and through A'; hence, it is an elation S'' and belongs to the group $H_1'(Al)$. Evidently, all elations in $H_1(Al)$ are contained in $G_3(S)$.

The remaining formulæ are easily verified.

§2. Singular Transformations.

We come to the consideration of the so-called singular transformations, Art. 318, in the collineation groups of the plane. These were defined as systems of collineations of one type not forming a continuous group, yet occurring in an otherwise continuous group of another type. We shall find two distinct kinds of singular transformations, viz.: discrete systems of collineations of type III or V occurring in groups of type II, second class; and discrete systems of collineations of type II occurring in groups of type I, second and third classes.

We shall examine our systems of singular transformations to see if they have both group properties; we shall find that the systems of singular transformations of types III and V occurring in groups of type II, second class, have both group properties, and hence are discontinuous subgroups of continuous groups. We shall also find that the singular transformations of type II occurring in groups of type I, second and third classes, have the second group property but not the first, hence they do not form discontinuous subgroups of continuous groups.

322. *Singular Transformations in* $G'_3(Al')_a$. Take two collineations of type II, $T'(AA'l')$ and $T'(AA_1'l')$, belonging to the group $G_3'(Al')_a$ and let their constants k and t be (a^t, t) and (a^{t_1}, t_1). For every value of $a \neq 0$ or ∞, t and t_1 may be so chosen that $t + t_1 \neq 0$, while $a^{t+t_1} = 1$. To prove this, put $a = re^{i\theta}$ and $t + t_1 = p + iq$; we then have

$$(re^{i\theta})^{(p+iq)} = 1. \tag{7}$$

Taking logarithms of both sides we get

$$(p + iq)(\log. r + i\theta) = 2n\pi i;$$

whence $p \log. r - q\theta = 0$ and $q \log. r + p\theta = 2n\pi$. Solving for p and q we find

$$p = \frac{2n\pi\theta}{\log.^2 r + \theta^2}, \quad q = \frac{2n\pi \log. r}{\log.^2 r + \theta^2}. \tag{8}$$

n has only integral values and hence, for values of $a \neq 0$ or ∞, and $n \neq 0$, t and t_1 can always be chosen so as to satisfy the conditions

$$t + t_1 \neq 0 \text{ and } a^{t+t_1} = 1. \tag{9}$$

The group $G_3'(Al')_a$ has the following structure: $G_3'(Al')_a = \infty^1 G_2'(ll')_a + H_2'(l') +$ S. T. If $T'(AA'l')$ and $T_1'(AA_1'l')$ be so chosen that $t + t_1 = 0$, then $a^{t+t_1} = 1$; their resultant is identical along l' and parabolic through A; hence, it is an elation $S'(Xl')$ where X is some point on l. This elation belongs to the group $A_2'(l')$. All elations of the group $A_2'(l')$ are present in $G_2'(Al')_a$. But if T' and T_1' be so chosen that $t + t_1 \neq 0$ and $a^{t+t'} = 1$, then their resultant is parabolic along the invariant line l' and also parabolic through the invariant point A; hence, it is a collineation of type III, $T''(Al')$.

Now t and t_1 may be so chosen that $t + t_1 \neq 0$ while $a^{t+t_1} = 1$ in an infinite number of ways, one for each integral value of n in equations (8). Hence, there are ∞^1 such collineations T'' in $G_3'(Al')$. But since n can have only integral values, these collineations of type III do not form a continuous system.

323. *The Discontinuous Group $d\,G''(Al')$.* We shall now examine this discontinuous system of collineations and see if it has one or both of the defining group properties. Let $T''(Al')$ and $T_1''(Al')$ be two collineations of this system whose parabolic constants along l' are t and t_1. Their resultant is of type III, since both belong to the group $G_3''(Al')$. Since t and t_1 are both of the form

$$t = p + i\,q = \frac{2n\pi\,(\theta + i \log r)}{\theta^2 + \log^2 r} \tag{10}$$

where n has only integral values, we see that t_2, the sum of t and t_1, is of the same form and $n_2 = n + n_1$. Hence $T_2''(Al')$, the resultant of T'' and T_1'', belongs also to the discontinuous system, and this system of singular transformations in $G_3(Al')_a$ has the first group property.

Let T'' be one of the collineations of the discontinuous system whose parabolic constant along l is $t = \frac{2n\pi\,(\theta + i \log r)}{\theta^2 + \log^2 r}$. The parabolic constant of T''^{-1}, the inverse of T'', is $-t$ and hence is of the same form as t with $-n$ for n in (10). Hence, the inverse of every collineation in the system is also in the system; thus this system of singular transformations has the second group property.

Since this discrete system of singular transformations in $G_3'(Al')_a$ has both group properties, it is a group, but a discontinuous subgroup of $G_3'(Al')$. We shall designate it by $d\,G''(Al')$.

THEOREM 1. The system of singular transformations of type III in $G_3'(Al')$ forms a discontinuous group, $d\,G''(Al')$.

324. *Singular Transformations in $G_2'(ll')_a$.* If $T'(AA'l')$ and $T_1'(AA_1'l')$ be chosen from the same group $G_2'(ll')_a$ and such

that $t+t_1=0$ and $a^{t-t_1}=1$, their resultant is parabolic along l and identical along l'. The resultant is therefore an elation belonging to the group $H_1'(Al')$. Every collineation in the group $H_1'(Al')$ is contained in $G_2'(ll')_a$. But if T' and T_1' are so chosen that $t+t_1\neq 0$ and $a^{t+t_1}=1$, then their resultant is parabolic along both l' and l. The resultant is an elation but it does not belong to $H_1'(Al')$. The vertex of its invariant figure is at A and the axis of invariant points is a line through A not l or l'. Since n has only integral values, this system of elations is not continuous and does not form a continuous group. The inverse of every elation in the system is also in the system.

Since the group $G_2'(ll')_a$ is contained in the groups $G_3'(Al')_a$, its system of singular transformations of type V is also contained in $G_3'(Al')_a$; hence $G_3'(Al')_a$ contains singular transformations of two kinds, viz.: types III and V.

Since the group $G_3'(Al')_a$ contains ∞^1 subgroups $G_2'(ll')_a$, one for each line l through A, it must therefore contain a discrete system of ∞^2 elations $S'(A)$, selected from the group $H_2'(A)$, which system has both group properties and thus forms a discontinuous subgroup of $G_3'(Al')_a$. But $G_3'(Al')_a$ also has $H_2'(l')$ as a subgroup. If we combine an elation from $S'(A)$ with one from $H_2'(l')$, the resultant is a collineation $T''(Al')$ of type III. Hence the singular transformations of type III in $G_3'(Al')_a$ appear as the resultants of the ∞^2 elations in $S'(A)$ with those of the group $H_2'(l')$. We also see that the discontinuous group $S'(A)$ is a subgroup of the discontinuous group $d\,G''(Al')$.

THEOREM 2. The system of singular transformations of type V in $G_2'(ll')_a$ form a discontinuous group $dS'(A)$.

In the same manner it may be shown that the groups $G_3'(A'l)_a$ and $G_2'(AA')_a$ dualistic to $G_3'(Al')_a$ and $G_2'(ll')_a$ contain singular transformations as follows: $G_3'(Al')_a$ contains singular transformations of types III and V; $G_2'(AA')_a$ contains a system of singular transformations of type V.

THEOREM 3. The groups $G_3'(A'l)_a$ and $G_3'(Al')_a$ of type II, second class, each contains discrete systems of singular transformations of types III and V; the groups $G_2'(Al')_a$ and $G_2'(AA')_a$ each contains discrete systems of singular transformations of type V.

325. *Singular Transformations in* $G_2(AA'l')_r$. We shall next examine the group $G_2(AA'l')_r$ for singular transformations. Its structure was given in Art. 320, thus: $G_2(AA'l')_r = \infty^1 G_1(AA'\overline{A''})_r + H_1'(Al) + S.\ T$. There are four cases to be considered, viz.: When r is rational with even numerator and odd denominator, r rational with odd numerator and odd denominator, r rational with odd numerator and even denominator, r irrational. We must examine each case separately.

Let r be rational with even numerator and odd denominator. Then according to article 256 each subgroup $G_1(AA'A'')_r$ in $G_2(AA'l')_r$ contains the same involutoric perspective collineation S, having its vertex at A' and its axis along l'; and the group $G_2(AA'l')_r$ contains only this one involutoric perspective collineation S.

Let S be combined with T any collineation of type I in $G_2(AA'l')_r$; the resultant one-dimensional transformations along both l and l' are both loxodromic, and hence the resultant of S and T is of type I and is one of the collineations of the group $G_2(AA'l')_r$. On the other hand, let S be combined with S', a collineation of type V in $H_1'(Al)$ (which is a subgroup of $G_2(AA'l')_r$). Along l we have an involutoric combined with an identical transformation, and the resultant in this direction is involutoric with invariant points at A and A'. Along l' we have an identical combined with a parabolic transformation, and the resultant is parabolic. Through A' we have an identical combined with a parabolic transformation and the resultant is also parabolic. Hence, the resultant of S and S' is T', a collineation of type II, whose invariant figure is $(AA'l')$ and whose constants k and t are $k = -1$ and t equal to the t of S. If S be combined in turn with each elation of $H_1'(Al)$, we have an infinite system of collineations of type II, for each of which $k = -1$ while t has all complex values. The parameters k and t of this system of collineations of type II do not satisfy the relation $k = a^t$, and hence (article 258) do not form a continuous group. These collineations of type II in $G_2(AA'l')_r$ are singular transformations. The structure of $G_2(AA'l')_r$ is thus seen to be

$$G_2(AA'l')_r = \infty\, G_1(AA'A'')_r + H_1'(Al) + \infty^1 T'.$$

Let us next examine for singular transformations the group $G_2(AA'l')_r$, where r is rational with odd numerator and even denominator. Each subgroup $G_1(AA'A'')_r$ of $G_2(AA'l')_r$ contains one perspective involutoric collineation S; these are all different, and form a system (S) whose common axis is l and whose vertices are in turn every point on l'. Any one of these perspective collineations, combined with a collineation of type I in $G_2(AA'l')_r$, results in a collineation of type I also belonging to $G_2(AA'l')_r$. The resultant of any collineation of the system (S) with any elation of the group $H_1'(Al)$ is another perspective collineation S_1 of the system (S). The resultant of any two perspective collineations of the system (S) is an elation belonging to the group $H_1'(Al)$. Thus the group $G_2(AA'l')_r$, where r is rational with odd numerator and even denominator, contains ∞^1 involutoric perspective collineations, but no singular transformations of type II. These perspective collineations are not singular transformations in the sense of the definition, for each of them belongs to a subgroup of $G_2(AA'l')_r$.

In like manner, it may be shown that the group $G_2(AA'l')_r$, where r is rational with odd numerator and odd denominator, contains ∞^1 involutoric perspective collineations having a common vertex at A and axes in turn every line through A'; but it contains no singular transformations. The group $G_2(AA'l')_r$, where r is irrational, contains (Art. 256) no involutoric perspective collineation and hence no singular transformations.

From these results, we conclude that the group $G_2(AA'l')_r$ contains singular transformations of type II when, and only when, its subgroups have one common involutoric perspective collineation S. This depends on the manner in which the invariant triangles of the subgroups $G_1(AA'A'')_r$ are put together. Thus, when the line l' passes through A, the group $G_2(AA'l')_r$ contains singular transformations when r has even numerator and odd denominator. On the other hand, if l' passes through A', in which case it is designated by l'', the group $G_2(AA'l'')_r$ has singular transformations when r has odd numerator and even denominator.

THEOREM 4. The group $G_2(AA'l')_r$ where r is a rational number such that the subgroups of $G_2(AA'l')_r$ all contain the same involutoric perspective collineation S, contains singular transformations; these are of type II, and all have the same value of k, viz.: $k=-1$.

326. *Other Groups Containing Singular Transformations of Type II.* Any collineation group of the plane which contains no subgroup of type II and which contains subgroups of the variety $G_2(AA'l')_r$ such that each contains but a single perspective involutoric collineation S, will evidently contain singular transformations of type II. In addition to $G_2(AA'l')_r$, the following groups of type I, second class, also contain singular transformations of type II: $G_3(AA')_r$, $G_3(ll')_r$, $G_4(Al)_r$.

In type I, third class, the group $G_3(AlS)$ contains ∞^1 subgroups $G_2(AA'l')_{r=2}$. Each of these subgroups contains ∞^1 singular transformations, and hence $G_3(AlS)$ contains ∞^2 singular collineations of type II.

The group $G_3(A, l'')_{r=-1}$ contains ∞^1 subgroups $G_2(AA'l'')_{r=-1}$. There is evidently but one involutoric perspective collineation S in the group $G_3(A, l'')_{r=-1}$, and this has its vertex at A and its axes coinciding with l''; S belongs to every subgroup $G_2(AA'l'')_{r=-1}$ in $G_3(Al'')_{r=-1}$. S, combined in turn with each of the ∞^1 elations in $G_3(A, l'')$, gives ∞^2 singular transformations of type II.

The group $G_5(l'')_{r=-1}$ contains ∞^2 subgroups $G_3(A, l'')_{r=-1}$, one for each position of A in the plane; hence, it contains ∞^3 subgroups $G_2(AA'l'')_{r=-1}$. Since each of these two-parameter subgroups contains ∞^1 singular transformations of type II, it follows that the group $G_5(l'')_{r=-1}$ contains ∞^4 such singular transformations.

The group $G_5(A)_{r=-1}$, dualistic to $G_5(l'')_{r=-1}$, also contains ∞^4 singular transformations of type II.

THEOREM 5. The following groups, and no others, contain singular transformations of type II: $G_2(AA'l')_r$, $G_3(AA')_r$, $G_3(ll')_r$, and $G_4(Al)_r$ (when r is rational); $G_3(AlS)$, $G_3(A,l'')_{r=-1}$, $G_5(l'')_{r=-1}$ and $G_5(A)_{r=-1}$.

§3. Mixed Groups.

327. In this section the following problems will be investigated: To find (1) all collineations in the plane that leave fixed one vertex of a triangle and interchange the other two vertices; (2) that interchange a pair of points; (3) that interchange a pair of lines; (4) that permute the vertices of a triangle.

These four problems lead us to the consideration of certain *mixed* groups of collineations. A mixed group is defined as a system of collineations which has both group properties and which is composed of a continuous group and certain discontinuous groups, which interchange certain parts of the invariant figure of the continuous group. For example, all collineations, leaving a triangle invariant as a whole, form a mixed group $m\,G(AA'A'')$, which consists of the continuous group $G_2(AA'A'')$ and all other collineations interchanging a pair of its vertices and also those permuting the three vertices.

In order to determine all the mixed groups of collineations in the plane, we must first examine all varieties of invariant figures of reducible groups of collineations. There are only eight varieties of such figures (Fig. 24), viz.: A point (A), a line (l), a lineal element (Al), a pair of points (AA'), a pair of lines (ll'), a point and line not incident (A, l), two points and two lines $(AA'l')$, and a triangle $(AA'A'')$. Of these eight only three may be the invariant figures of mixed groups, viz.: (AA'), (ll') and $(AA'A'')$, for these are the only figures in which points or lines may be interchanged without changing the figures as a whole.

328. *The Mixed Group* $mG_2(APQ)$. Let A, P, Q be any three points forming a triangle. Any collineation that interchanges P and Q must leave the line PQ invariant. Let $(AA'A'')$ be the invariant triangle of a collineation of type I, which interchanges P and Q; then two of the vertices, as A' and A'', must be on the line PQ and so situated that the cross-ratio $(A'A''PQ) = -1$. All collineations in the group $G_2(AA'A'')$, for which the cross-ratio along the side $A'A''$ is 1, leave A invariant and interchange P and Q; they belong, therefore, to the mixed group $mG_1(APQ)$.

Let k and k' be the two independent parameters of the continuous group $G_2(AA'A'')$. Since the product of the three cross-ratios in the same order around the triangle must be unity, we have $(k)(-1)(1/k') = 1$; thus, $k + k' = 0$. Hence, out of the ∞^2 collineations in $G_2(AA'A'')$, where $(A'A''PQ) = -1$, there are ∞^1 that satisfy the condition $k + k' = 0$ and interchange P and Q. The pair of points $A'A''$ can be chosen in ∞^1 different ways, so that $(A'A''PQ) = -1$; if out of each of these groups $G_2(AA'A'')$ we select the collineations that satisfy the relation $k + k' = 0$, we obtain ∞^2 collineations of type I that leave A invariant and interchange P and Q.

Among these ∞^2 collineations of type I interchanging P and Q, there are ∞^1 of type IV. In every group $G_2(AA'A'')$ there are two collineations, viz.: $k = 1, k' = -1$ and $k = -1, k' = 1$ which satisfy the condition $k + k' = 0$ and are not of type I. They are involutoric perspective collineations with the vertex always on the line PQ.

Since no collineation of type III or V is involutoric along an invariant line, it follows that the mixed group $mG_2(APQ)$ contains no collineations of these types. A collineation of type II, whose invariant figure is $(AA'l')$, may be involutoric along AA', but cannot belong to the mixed group $mG_2(APQ)$, for the point A would then have to be a second invariant point on the line l', which is impossible.

THEOREM 6. The mixed group $mG_2(APQ)$ contains, in addition to the continuous group $G_2(APQ)$, ∞^2 collineations of type I and ∞^1 of type IV, which leave A invariant and interchange P and Q.

329. *The Mixed Groups* $mG_4(PQ)$ *and* $mG_4(ll')$. The continuous group $G_4(PQ)$ contains ∞^4 collineations leaving P and Q separately invariant; we seek, in addition to these, all collineations which interchange P and Q. There are ∞^3 triangles $(AA'A'')$ so situated that $A'A''PQ$ are collinear and the cross-ratio $(A'A''PQ) = -1$. Each of these triangles is the invariant triangle of a two-parameter group in which ∞^1 collineations satisfy the relation $k + k' = 0$, and hence interchange P and Q. Therefore, there are ∞^4 collineations of type I which interchange P and Q.

Let us consider the groups $G_2'(A'A''l')$ of type II, where $(A'A''PQ) = -1$. The collineations of this group depend upon two parameters k and t. When $k = -1$, the transformations along $A'A''$ are involutoric and interchange P and Q. The group $G_2'(A'A''l')$ contains ∞^1 such collineations, one for each value of t. The figure $(A'A''l')$ can be chosen in ∞^2 different positions satisfying the condition $(A'A''PQ) = -1$. Hence, there are ∞^3 collineations of type II which interchange P and Q.

Let g be any line of the plane cutting PQ in A' and take A'' on PQ such that $(A'A''PQ) = -1$. The involutoric collineation of the group $H_1(A'', g)$ interchanges P and Q. There are evidently ∞^2 such involutoric collineations, one for each line of the plane not passing through P or Q.

In like manner, it may be shown that the mixed group $mG_4(ll')$ has a similar structure to $mG_4(PQ)$; these groups are dualistic, and the properties of the former may be inferred at once from those of the latter.

THEOREM 7. The mixed group $\left\{\begin{matrix} mG_4(PQ) \\ mG_4(ll') \end{matrix}\right\}$ contains, besides the continuous group $\left\{\begin{matrix} G_4(PQ) \\ G_4(ll') \end{matrix}\right\}$, ∞^4 collineations of type I, ∞^3 of type II and ∞^2 of type IV, which interchange $\left\{\begin{matrix} P \\ l \end{matrix}\right\}$ and $\left\{\begin{matrix} Q \\ l' \end{matrix}\right\}$.

330. *Collineations which Permute the Vertices of a Triangle.* Let P, Q, R be the vertices of a triangle; we wish to find all collineations which change P into Q, Q into R and R into P; also, their inverses, viz.: those that change P into R, R into Q and Q into P. Let K be any conic circumscribing the triangle; with PQR as the triangle of reference, the homogeneous equation of K may be written

$$\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0. \tag{11}$$

Let x', y', z' be the coordinates of any point A; the polar of A with respect to the triangle of reference is

$$\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0. \tag{12}$$

The polar of A with respect to the conic K is given by

$$x(bz' + cy') + y(az' + cx') + z(ay' + bx') = 0. \tag{13}$$

We wish to determine the point x', y', z', so that its polars with respect to the triangle (PQR) and the conic K coincide. Comparing equations (12) and (13), we find $x' : y' : z' = a : b : c$.

When the point A is not on a side of the triangle of reference, we find one, and only one, position of A such that its polar with respect to the triangle is at the same time its polar with respect to the conic K. The converse of this proposition is also true; if we choose any point A and take its polar l with respect to the triangle of reference, we shall find one, and only one, conic K circumscribing the triangle for which A and l are pole and polar.

The line l and conic K, whose equations are respectively

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 0 \text{ and } \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 0,$$

intersect in a pair of points A' and A''. $(AA'A'')$ is the invariant triangle of a two-parameter group $G_2(AA'A'')$; the path-curves of its one-parameter subgroup $G_1(AA'A'')_r$

are conics and K is one of these path-curves. Consequently, all collineations of the group $G_1(AA'A'')_{r=-1}$ transform K into itself.

The coordinates of A are (a, b, c) of A' $(a, \omega b, \omega^2 c)$, of A'' $(a, \omega^2 b, \omega c)$, where $\omega^3 = 1$. The lines AA', AA'', AP, AQ, AR are given by the following equations respectively:

$$\frac{x}{a} + \frac{y}{\omega^2 b} + \frac{z}{\omega c} = 0, \quad \frac{x}{a} + \frac{y}{\omega b} + \frac{z}{\omega^2 c} = 0,$$
$$\frac{y}{b} = \frac{z}{c}, \quad \frac{x}{a} = \frac{z}{c}, \quad \frac{x}{a} = \frac{y}{b}. \tag{14}$$

From these equations we readily find that the cross-ratios of the three pencils $A(A'A''PQ)$, $A(A'A''QR)$, $A(A'A''RP)$ are equal to each other and each equal to ω, *i. e.*, $e^{\frac{2\pi i}{3}}$. Consequently, the collineations of the group $G_1(AA'A'')_{r=-1}$, for which $k = e^{\frac{2\pi i}{3}}$ changes P into Q, Q into R, and R into P. The inverse of this collineation, for which $k = e^{\frac{4\pi i}{3}}$, changes P into R, R into Q, and Q into P.

There are ∞^2 conics circumscribing the triangle PQR; for each of these conics there is a point A and a line l which are pole and polar with respect to both triangle and conic. Consequently, there are ∞^2 two-parameter groups $G_2(AA'A'')$, each of which contains a pair of inverse collineations that permute the vertices of the triangle (PQR). Each of these collineations is of order 3.

THEOREM 8. There are ∞^2 collineations of type I and period 3 that permute the vertices of a triangle.

331. *The Mixed Group $mG_2(AA'A'')$.* The mixed group $mG_2(AA'A'')$ is composed of all collineations which leave the triangle $(AA'A'')$ invariant. These consist of the ∞^2 collineations belonging to the continuous group $G_2(AA'A'')$, the ∞^2 collineations of type I which leave one vertex fixed and interchange the other two, the ∞^2 collineations of type I and period 3 which permute the vertices of the triangle, and of the ∞^1 collineations of type IV which leave one vertex invariant and interchange the other two.

§4. Generation of Finite from Infinitesimal Collineations.

332. In the theory of continuous groups as developed by Prof. Sophus Lie, the infinitesimal transformation plays the most important part. The generation of finite transformations and whole groups of transformations by the repetition of infinitesimal transformations of the group, is a fundamental part of his theory. In the theory of continuous groups of collineations in one and two dimensions, developed in the preceding pages, the infinitesimal collineation plays no such important role. It is, however, of prime importance for us to know in what manner, subject to what conditions, and under what limitations, the finite collineations of a continuous group may be generated by the repetition of infinitesimal collineations of the group. In this way the points of contact of the present theory with Lie's theory will be most forcibly exhibited. The theory of the generation of real continuous groups of collineations from real infinitesimal collineations differs so markedly from that of the generation of complex groups from complex infinitesimal collineations that the two cases are best treated separately.

333. *A. Generation of Complex Groups.* It has been shown in the previous pages that every n-parameter group of collineations $n < 8$, is composed of one-parameter subgroups, and that every one-parameter group contains at least one infinitesimal collineation. Our immediate problem is to discuss the generation of one-parameter groups from their infinitesimal collineations. We found in one dimension two types of one-parameter groups, viz.: $G_1(AA')$ and $G_1'(A)$. In two dimensions we found five types of one-parameter groups, viz.: $G_1(AA'A'')_r$, $G_1'(AA'l)_a$, $G_1''(AlS)$, $H_1(A,l)$ and $H_1'(Al)$. Each of these types must be discussed separately.

334. *The One-dimensional Parabolic Group* $G_1'(A)$. It was shown in Chap. I, Art. 28, that the variable parameter of

the group $G_1'(A)$ is t, and that the law of combination of parameters in this group is expressed by the equation $t_2 = t + t_1$. The transformations in this group are commutative, Art. 11, and t assumes in turn all complex values.

The group contains the identical transformation corresponding to the value $t = 0$. By definition, Art. 26, an infinitesimal transformation is one that differs by an infinitesimal value of the parameter from the identical transformation. Let us write t in the form $re^{i\theta}$, where r is real and positive. The identical transformation of the group is given by $r = 0$. Infinitesimal transformations of the group are given by $r = \delta$, where δ is an infinitesimal. Since θ varies continuously from 0 to 2π, the relation $t = \delta e^{i\theta}$ gives us an infinite number of infinitesimal transformations in the group $G_1'(A)$. If t be represented geometrically by the Argand diagram, the values of t corresponding to these ∞^1 infinitesimal transformations lie on a circle about the origin of radius $r = \delta$.

Let us choose one of these infinitesimal transformations corresponding to a fixed value of θ, say θ_1, and designate it by I_1. If I_1 be repeated we find the resultant of I_1 and I_1 as follows: $t_2 = t_1 + t_1 \equiv \delta e^{i\theta_1} + \delta e^{i\theta_1} = 2\delta e^{i\theta_1}$. In like manner if I_1 be repeated n times, we have $t_n = n\delta e^{i\theta_1}$. The position of the point t_n on the Argand diagram is at a distance $n\delta$ from the origin and on a line making the angle θ_1 with the axis of reals. By a proper choice of n we can make the point t_n move from the origin along the half-ray l_{θ_1} to any desired position on this ray. Consequently every transformation in $G_1'(A)$ corresponding to a value of t situated on the half-ray l_{θ_1} can be generated by the repetition of the infinitesimal transformation I_1 given by $t_1 = \delta e^{i\theta_1}$. In like manner we see that each infinitesimal transformation in $G_1'(A)$ can generate those finite transformations of the group whose corresponding values of t lie in the Argand diagram on a half-ray through the origin. It is evident that any given finite transformation in $G_1'(A)$ can be generated by the repetition of one, and only one, infinitesimal transformation.

THEOREM 9. The group $G_1'(A)$ contains ∞^1 infinitesimal transformations; every finite transformation in the group can be generated from one, and only one, infinitesimal transformation of the group.

335. *The One-dimensional Loxodromic Group* $G_1(AA')$. It was shown in Art. 26 that the variable parameter of the loxodromic group $G_1(AA')$ is k, a complex number, and that the law of combination of parameters in this group is expressed by the equation $k_2 = k k_1$. In this group as in $G_1'(A)$ the transformations are commutative and k assumes in turn all complex values.

This group contains the identical transformation corresponding to the value $k = 1$. When the values of k are represented on the Argand diagram, the unit point, $k = 1$, corresponds to the identical transformation. Each point on the circle about the unit point with radius δ, an infinitesimal, corresponds to an infinitesimal transformation of the group. Hence the group $G_1(AA')$ contains ∞^1 infinitesimal transformations.

Let us set $k = r e^{i\theta}$ and $r = e^{c\theta}$; whence $k = e^{(c+i)\theta}$, where c is a real number. When $\theta = 0$, $k = 1$ for all finite values of c. When $\theta = \delta\theta$, an infinitesimal, we have an infinitesimal transformation corresponding to each finite value of c. Let us choose one of these infinitesimal transformations, say that corresponding to the fixed value, c_1, and designate it by I_1. If I_1 be repeated n times we have $k_n = e^{(c_1+i)n\delta\theta}$. By choosing n sufficiently large we may thus generate from I_1 certain finite transformations of the group. The locus of the point $k_n = e^{(c+i)n\delta\theta}$, as n varies, is a logarithmic spiral about the origin, passing through the unit point and making an angle ψ with the axis of reals such that $\cot\psi = c$.

Such a spiral makes an infinite number of turns about the origin and the unit point divides it into two distinct portions, which we shall call the two halves of the spiral. One of these halves lies entirely within the unit circle and the other entirely without it. This spiral contains two points which correspond to infinitesimal transformations, one in each half of the spiral and adjacent to the unit point. These are given by $k = e^{+(c_1+i)\delta\theta}$ and $k = e^{-(c_1+i)\delta\theta}$. Every finite transformation corresponding to a point on either half of the spiral can be generated by the repetition of its corresponding infinitesimal transformation.

Different values of c give us different spirals. c varies continuously through all real values from $-\infty$ to $+\infty$, so that these spirals lie infinitely close to one another. They all pass through the unit point. As c approaches zero, the corresponding spiral approaches as a limit the circle of unit radius about the origin; as c approaches infinity, the corresponding spiral approaches as a limit the straight line which is the axis of reals.

Two problems now present themselves for solution: Can every finite transformation in the group $G_1(AA')$ be generated by the repetition of an infinitesimal transformation of the group? Can a given finite transformation T of the group be generated by more than one infinitesimal transformation of the group? To answer these questions we proceed as follows: Let P be the point on the Argand diagram corresponding to the given transformation T_{k_1} and let the coordinates of P be ρ_1 and $\theta_1 + 2\pi n$ (n any integer). Since

$$k_1 = \rho_1 e^{i(\theta_1+2n\pi)} = e^{(c+i)\,(\theta_1+2n\pi)}, \text{ then}$$

$$\log k_1 \equiv \log \rho_1 + i(\theta_1 + 2n\pi) = c(\theta_1 + 2n\pi) + i(\theta_1 + 2n\pi)\,;$$

whence $\log \rho_1 = c(\theta_1 + 2n\pi)$ or $c = \frac{\log \rho_1}{\theta_1 + 2n\pi}$. Since n is any integer, there are an unlimited number of values of c which satisfy the equation. Thus there are an infinite number of spirals of the family $\rho = e^{(c+i)\theta}$ through the point P. When $n = 0, 1, 2, 3, \ldots$ the corresponding spiral starting from the unit point, makes $0, 1, 2, 3, \ldots$ turns about the origin before passing through P. Hence, every point in the plane of the Argand diagram, *not on the unit circle,* lies on an infinite number of discrete spirals, from which we infer that every transformation of the group $G_1(AA')$, whose corresponding point in the Argand diagram does not lie on the unit circle, can be generated from an infinite number of distinct infinitesimal transformations of the group. If the point P lies on the unit circle about the origin, the corresponding transformation T may be generated from either of the two infinitesimal transformations, $k = e^{+i\delta\theta}$ or $k = e^{-i\delta\theta}$.

The solutions of our two problems are stated in the following theorem:

THEOREM 10. Every finite transformation of the group $G_1(AA')$ can be generated by the repetition of an infinitesimal transformation of the group; every finite transformation T of the group for which $k = re^{i\theta}$, $r \neq 1$, can be generated from an infinite number of discrete infinitesimal transformations of the group: if $r = 1$, T can be generated from only two infinitesimal transformations of the group.

336. *The Two-dimensional Groups* $H_1'(Al)$, $G_1''(AlS)$ *and* $G_1(AA'l)_a$. Having discussed in detail the generation from infinitesimal transformations of finite one-dimensional projective transformations, we turn now to apply these results to the generation of finite collineations in two dimensions from infinitesimal collineations.

It was shown in articles 247, 269 and 261, that the three groups $H_1'(Al)$, $G_1''(AlS)$ and $G_1'(AA'l)_a$, all have the same structure; *i. e.*, in each of these groups the parameter is t and the law of combination of parameters is $t_2 = t + t_1$; t assumes in turn all complex values. It is evident at once that the results obtained for the one-dimensional group $G_1'(A)$ hold also for each of the two-dimensional groups $H_1'(Al)$, $G_1''(AlS)$ and $G_1'(AA'l)_a$. Each of these groups contains the identical transformation and an infinite number of infinitesimal transformations. Each of these infinitesimal transformations is the generator of those finite transformations of the group whose corresponding points on the Argand diagram lie on its half-ray through the origin. Every finite transformation in one of these groups can be generated from one, and only one, infinitesimal transformation of the group.

337. *The Groups $H_1(A, l)$ and $G_1(AA'A'')_r$.* It was shown in Arts. 245 and 250 that the two groups $H_1(A, l)$ and $G_1(AA'A'')_r$ have the same structure as the one-dimensional group $G_1(AA')$. In each group the parameter is k, which assumes in turn all complex values, and the law of combination of parameters is expressed by the equation $k_2 = kk_1$. Hence all the results obtained above for the group $G_1(AA')$ apply immediately to each of the groups $H_1(A, l)$ and $G_1(AA'A'')_r$. Each of these groups contains the identical collineation and an infinite number of infinitesimal collineations. Each of these infinitesimal collineations is the generator of those finite collineations of the group whose corresponding points on the Argand diagram lie on its half-spiral through the unit point. Every finite collineation in one of these groups can be generated from an infinite number of infinitesimal collineations (except those whose corresponding points on the Argand diagram lie on the unit circle); each of these exceptional collineations can be generated from two and only two infinitesimal collineations of the group.

338. *r-Parameter Groups of Plane Collineations.* The structure of all collineation groups of the plane was discussed in §§ 1 and 2 of the present chapter. In regard to structure the groups of plane collineations may be divided into two classes, viz.: those which do, and those which do not, contain singular transformations.

Groups containing no singular transformations are made up of one-parameter subgroups, such that every collineation in such a group belongs to at least one one-parameter subgroup. Consequently every finite collineation in an r-parameter group, G_r, which contains no singular collineations, can be generated from one or more infinitesimal collineations of the group G_r.

In groups which contain singular collineations it is evident that all non-singular collineations of the group can be generated from one or more infinitesimal collineations of the group; but no singular collineations in such a group can be generated from an infinitesimal collineation belonging to the group.

Every finite collineation in the group G_8, the group of all plane collineations, belongs to at least one one-parameter subgroup and hence can be generated from one or more infinitesimal collineations of the plane.

THEOREM 11. Every finite collineation in the plane belongs to at least one one-parameter subgroup of G_8 and can be generated from one or more infinitesimal collineations.

B. GENERATION OF REAL GROUPS.

We turn now to the question of the generation of real collineations in one and two dimensions from real infinitesimal collineations. We shall first discuss the question in one dimension. The real group RG_3 contains three types of one-parameter subgroups: viz., $pG_1(A)$, $eG_1(AA')$ and $hG_1(AA')$, which require separate consideration.

339. *The Group* $pG_1(A)$. The parabolic group $pG_1(A)$ contains the identical transformation for which $t = 0$ and two infinitesimal transformations for which $t = \pm\delta$, where δ is a real infinitesimal. Since $t_2 = t + t_1$, we see that every finite transformation in $pG_1(A)$, for which t is positive, can be generated from the positive infinitesimal transformation of the group; in like manner every finite transformation in $pG_1(A)$, for which t is negative, can be generated from the negative infinitesimal transformation of the group. This reasoning applies to every real parabolic transformation group of one-dimensional projective transformations.

THEOREM 12. Every real parabolic projective transformation in one dimension can be generated from one and only one real infinitesimal transformation.

340. *The Group* $eG_1(AA')$. The parameter k of the elliptic group $eG_1(AA')$ is of the form $k = e^{i\theta}$, where θ varies from $-\infty$ to $+\infty$. This group contains the identical transformation for which $\theta = 0$ and two infinitesimal transformations for which $\theta = \pm\delta$. Every finite transformation T

in the group may be generated from either of the infinitesimal transformations of the group. Indeed, since $e^{i\theta}$ is a periodic function of period 2π, T may be generated an infinite number of times from each infinitesimal transformation of the group.

THEOREM 13. Every real elliptic projective transformation in one dimension can be generated from two distinct infinitesimal transformations.

341. *The Group $hG_1(AA')$.* The parameter k of the hyperbolic group $hG_1(AA')$ is real and varies from $-\infty$ to $+\infty$. This group contains the identical transformation corresponding to $k=1$ and two infinitesimal transformations corresponding to the values $k=1\pm\delta$ where δ is an infinitesimal. Since $k_2=kk_1$, it follows that every finite transformation of the group, for which k is positive and greater than unity, can be generated from the infinitesimal transformation $k=1+\delta$; every transformation, for which k is positive and less than one, can be generated from the other infinitesimal transformation $k=1-\delta$; the transformation of the group for which k is negative can not be generated from either infinitesimal transformation of the group.

We thus see that the group $hG_1(AA')$ is composed of three subdivisions as follows: All transformations for which $1<k<\infty$ form subdivision I, and are generated from $k=1+\delta$; all for which $0<k<1$ form subdivision II, and are generated from $k=1-\delta$; all for which $-\infty<k<0$ form subdivision III and can not be generated from any real infinitesimal transformation.

THEOREM 14. Every real hyperbolic projective transformation in one dimension, for which k is positive, can be generated from one and only one real infinitesimal transformation; no real hyperbolic transformation with negative k can be generated from a real infinitesimal transformation.

342. *Real Collineations in the Plane.* We now take up the question of the generation of real finite plane collineations from real infinitesimal collineations. We must exam-

ine separately the five different types of plane collineations. We shall easily dispose of types V, IV and III, but it will be necessary to treat types I and II at greater length.

343. *Type V.* There are ∞^4 real collineations of type V which readily fall into ∞^3 one-parameter groups so that each of these finite collineations belongs to one and only one such subgroup. Hence we need only to discuss the generation of the finite collineations in one subgroup, say $RH_1'(Al)$, from the infinitesimal collineations of the group. The parameter of the group $RH_1'(Al)$ is t, which varies from $-\infty$ to $+\infty$, hence the structure of $RH_1'(Al)$ is precisely the same as that of $pG_1(A)$. We may therefore apply the results found above for $pG_1(A)$ directly to $RH_1'(Al)$. The group of elations $RH_1'(Al)$ contains two infinitesimal elations corresponding to the two values of $t = \pm \delta$; each of these infinitesimal collineations generates its corresponding subdivision of the group. The general statement may now be made as follows:

THEOREM 15. Each real collineation of type V may be generated from one and only one real infinitesimal collineation.

344. *Type IV.* There are ∞^5 real collineations of type IV in the plane which fall into ∞^4 one-parameter subgroups $RH_1(A', l)$, so that each perspective collineation of this type belongs to one and only one such subgroup. An examination of one of these subgroups, say $RH_1(A, l)$, shows that it is identical in structure with the group $hG_1(AA')$. Hence we may formulate the results immediately.

THEOREM 16. Each real collineation of type IV, for which k is positive, may be generated from one and only one real infinitesimal collineation; no such collineation, for which k is negative, can be generated from a real infinitesimal collineation.

345. *Type III.* There are ∞^6 real collineations of type III in the plane which fall into ∞^3 three-parameter groups of the kind $RG_3''(Al)$ in such a way that every collineation of type III belongs to one and only one such group. Each group $RG_3''(Al)$ breaks up into ∞^2 one-parameter subgroups

$RG_1''(AlS)$ in such a way that each collineation of type III in $RG_3''(Al)$ belongs to one and only one such subgroup. Hence each real collineation of type III in the plane belongs to one and only one one-parameter group, $RG_1''(AlS)$. The group $RG_1''(AlS)$ has exactly the same structure as the group $pG_1(A)$; consequently we may state our theorem at once.

THEOREM 17. Each real collineation of type III in the plane can be generated from one and only one real infinitesimal collineation.

346. *Type I.* There are two distinct kinds of real collineations of type I in the plane, viz.: hyperbolic and elliptic, article 310. These fall into ∞^0 two-parameter subgroups of RG_8: viz., $hG_2(AA'A'')$ and $eG_2(AA'A'')$ in such a way that each real collineation of type I belongs to one and only one of these subgroups. The two cases must be treated separately and we take up first the hyperbolic group $hG_2(AA'A'')$.

347. *The Hyperbolic Case.* The two-parameter group $hG_2(AA'A'')$ has for parameters k and k' both of which assume in turn all real values.

The two-parameter group $hG_2(AA'A'')$ contains an infinite number of one-parameter subgroups, and we proceed to determine these. All transformations in $hG_2(AA'A'')$ for which the two parameters satisfy a relation of the form $k' = k^r$, where r is a constant, form a one-parameter subgroup; and conversely, in all one-parameter subgroups, k and k' satisfy a relation of this form. There are different subgroups for different values of r. Geometrically, article 252, r is interpreted as the constant cross-ratio of certain four points on the tangent to a path curve of $hG_1(AA'A'')_r$: viz., the point of tangency T and the points of intersection of the tangent with the sides of the invariant triangle. These four points are all real for a real hyperbolic group and hence r is also real.

In order to study the distribution of the ∞^2 collineations of $hG_2(AA'A'')$ into one-parameter subgroups we resort to a geometrical device as follows: Let k and k' be the rectangular coordinates of a point in a plane (not to be confused with the plane of our transformation). It is evident, since k and k' are independent parameters, that there is a point in the plane corresponding to every collineation of the group $hG_2(AA'A'')$. Since all collineations, whose parameters k and k' satisfy the relation $k' = k^r$, form a one-parameter subgroup of $hG_2(AA'A'')$, we see that the curve $y = x^r$ corresponds to this subgroup and the individual points of the curve correspond to the individual collineations of the group. If we give to r all real values, we have a family of curves which corresponds to the system of subgroups of $hG_2(AA'A'')$.

From the properties of this system of curves we deduce the following results: If r is an irrational number, the curve $y = x^r$ contains no real point for which either coordinate is negative; the curve lies entirely in the first quadrant. If r is a rational fraction with even numerator and odd denominator, y can not be negative, and the curve lies above the axis of x in the first and second quadrants. If r is rational with odd numerator and even denominator, the curve lies in the first and fourth quadrants. If r is rational with odd numerator and odd denominator, the curve lies in the first and third quadrants.

Every curve passes through the point $(1, 1)$, which shows that the identical transformation belongs to every subgroup. The curves of our family contain every point in the first quadrant, but not every point in the second, third and fourth quadrants. Consequently our two-parameter group $hG_2(AA'A'')$ contains collineations *which do not belong to any of its subgroups.* Such a collineation has one or both of its cross-ratio constants negative, and their values are such that they do not satisfy an algebraic equation of the form of $k'^m = k^n$, where m and n are integers.

The variable parameter of every one-parameter group in $hG_2(AA'A'')$ is k; and every one-parameter group contains two real infinitesimal collineations: viz., when $k = 1 \pm \delta$. Each infinitesimal collineation generates its corresponding portion of the group. Every collineation in the group

$hG_2(AA'A'')$ for which both k and k' are positive can be generated from one and only one infinitesimal collineation of the group, while no collineation for which either k or k' is negative can be so generated.

THEOREM 18. Every real hyperbolic collineation of type I, for which k and k' are both positive, can be generated from one and only one real infinitesimal collineation: no such collineation, for which either k or k' is negative, can be generated from a real infinitesimal collineation.

348. *The Elliptic Case.* We turn now to the consideration of the real elliptic group $eG_2(AA'A'')$ in which the invariant triangle has one real vertex, A, and two conjugate imaginary vertices, A' and A''. It was shown in Art. 312, Chap. III, that k and k' are not independent parameters, but that they are conjugate imaginary numbers. Thus the real elliptic group $eG_2(AA'A'')$, instead of having two independent parameters k and k', has only one: viz., k; but this is a complex number and may assume in turn all possible complex values. Consequently the group $eG_2(AA'A'')$ contains a collineation corresponding to each point on the Argand diagram. Therefore the group $eG_2(AA'A'')$ has exactly the same structure as the one-dimensional loxodromic group $G_1(AA')$ discussed in Art. 27, and we may apply the results of that discussion directly to the present case.

The collineations forming the one-parameter subgroup $eG_1(AA'A'')$ correspond on the Argand diagram to the points on the logarithmic spirals $k = e^{(c+i)\theta}$ around the origin. A collineation T, corresponding to a point P not on the unit circle, belongs to an unlimited number of distinct subgroups and may be generated from an unlimited number of distinct infinitesimal collineations. If the point P, corresponding to T, lies on the unit circle, T belongs to only one subgroup and can be generated from either of two infinitesimal collineations: viz., $k = e^{\pm i\delta\theta}$.

THEOREM 19. Every real elliptic collineation of type I can be generated from either two or an unlimited number of real infinitesimal collineations.

349. *Type II.* The group $RG_2'(AA'l)$ contains two real parameters, k and t, each of which assumes in turn all real values. The group contains ∞^1 one-parameter subgroups for which k and t satisfy the relation $k = a^t$, Art. 299, where a is a constant. There is one such subgroup for each *positive* value of a.

In order to study the distribution of the transformations in $RG_2'(AA'l)$ into subgroups and their generation from infinitesimal transformations, we resort to the same device as in type I, and make k and t the rectangular coordinates of a point in a plane. The family of curves $y = a^x$ represents the system of one-parameter subgroups of $RG_2(AA'l)$. For positive values of a these curves lie in the first and second quadrants and completely fill the upper half of the plane. There are no continuous curves for negative values of a, and hence continuous subgroups of $RG_2(AA'l)$ exist only for positive values of a.

Two particular curves of the family deserve special attention. For a very large value of a, the curve $y = a^x$ differs but little from the axis $x = 0$; hence in the limit when $a = \infty$ the line $x = 0$ is a curve of the family; on the other hand when $a = 1$ the curve reduces to the line $y = 1$. In the first case $x = 0$ is the only curve of the family that penetrates into the lower half of the plane, and consequently the corresponding group is the only continuous subgroup of $RG_2(AA'l)$ containing collineations with negative values of k. The collineations of the group corresponding to $a = \infty$ are of type IV. The collineations of the group corresponding to $a = 1$ are of type V.

Each one-parameter subgroup of $RG_2(AA'l)$ contains two infinitesimal collineations, one positive and the other negative. Every collineation in a subgroup of $RG_2(AA'l)$ may be generated from one of its infinitesimal collineations, except the collineations with negative k in the perspective subgroup $a=\infty$. The collineations properly of type II in $RG_2(AA'l)$ for which k is negative *do not belong to its continuous subgroups* and can not be generated from infinitesimal collineations of the group.

THEOREM 20. Every real collineation of type II, for which k is positive, belongs to one and only one one-parameter group $RG_1(AA'l)_a$ and can be generated from one and only one real infinitesimal collineation; no real collineation of type II, for which k is negative, can belong to a one-parameter group nor can it be generated from a real infinitesimal collineation.

§ 5. The General Linear Group, $G_6(l_\infty)$.

350. In chapter IV we determined all varieties of subgroups of the general projective group G_8. Some of these groups are of special importance when the invariant figure of the group is especially related to the Absolute of the Euclidian plane, *i. e.*, to the line at infinity and the two circular points. In the present section we shall study in detail the general linear group, $G_6(l_\infty)$ and its subgroups: the special linear group, $G_5(l_\infty)$, or group of Invariant Areas, the group of Similarity, $G_4(\omega\omega')$, and the group of all Motions of a rigid body in the Euclidian plane, $G_3(\omega\omega')_{r=-1}$.

351. *Invariant Line at Infinity.* The six-parameter group of collineations whose equations are in the linear form

$$x_1=ax+by+c,\quad y_1=a'x+b'y+c', \tag{15}$$

is called the general linear group. In the general linear fractional transformation the line represented by the common denominator is transformed into the line at infinity (Art. 82). In the above form the denominator is a constant and represents the line at infinity, which is thus transformed into itself.

Equations (15) contain six constants or parameters. The first group property is shown at once by eliminating x_1 and y_1

from two transformations of the group, T and T_1, whose equations are (15) and (15_a):

$$x_2 = a_1 x_1 + b_1 y_1 + c_1; \quad y_2 = a_1' x_1 + b_1' y_1 + c_1'. \qquad (15_a)$$

The resultant is also a linear transformation. The second group property is shown by solving equations (1) for x_1 and y_1. The inverse of T is also linear and the second group property is established.

THEOREM 21. All linear transformations in two variables form a six-parameter group $G_6(l_\infty)$ whose invariant figure is the line at infinity.

352. *Parallel Lines are Transformed into Parallel Lines.* Since the line at infinity is an invariant line of the group $G_6(l_\infty)$ every point at infinity remains at infinity and hence parallel lines are transformed into parallel lines by every transformation of the group. The common direction of the system of parallel lines may be altered but the fact of parallelism is preserved.

This conclusion may be shown analytically as follows: Let the equation of a system of parallel lines be written

$$Ax_1 + By_1 + C = 0, \qquad (16)$$

where A and B are constants and C a variable parameter. Substitute for x_1 and y_1 their values in the equations

$x_1 = ax + by + c, \quad y_1 = a'x + b'y + c'$, and we get

$$x(Aa + Ba') + y(Ab + Bb') + Ac + Bc' + C = 0. \qquad (16)$$

Since the coefficients of x and y are constants and only C varies we have again a system of parallel lines.

353. *Parabolas are Transformed into Parabolas.* All conics touching the invariant line at infinity are transformed into conics touching the same invariant line; hence the transformations of the group $G_6(l_\infty)$ transform the system of ∞ parabolas of the plane into the same system of parabolas. The general equation of a parabola is

$$(\alpha x_1 + \beta y_1)^2 + 2gx_1 + 2fy_1 + d = 0. \qquad (17)$$

Substituting the values of x_1 and y_1 from equations (15) we get $[\alpha(ax+by+c)+\beta(a'x+b'y+c')]^2+2g(ax+by+c)+2f(a'x+b'y+c')+d=0$; this may be written in the form

$$[(\alpha a+\beta a')x+(\alpha b+\beta b')y]^2+2Gx+2Fy+C=0, \qquad (17_a)$$

which is again the equation of a parabola.

THEOREM 22. Parallel lines are transformed into parallel lines and parabolas into parabolas by all collineations of the group $G_6(l_\infty)$.

354. *All Areas are Altered by a Constant Ratio R.* One of the most important properties of group $G_6(l_\infty)$ is that any collineation T of the group changes all areas of the plane by a constant ratio R. For example, if Δ represents any area of the plane, it is transformed by T into a new area Δ_1, such that $\Delta_1=R\Delta$; where R is a function of the parameter of T only and independent of the position, shape or size of Δ.

To prove* this consider any triangle (ABC) and its corresponding triangle $(A_1B_1C_1)$. Draw lines through the vertices of (ABC) parallel to the opposite sides; three new triangles will be thus constructed; draw new lines through these vertices and so continue until the whole plane is divided into a net of equal triangles; do the same for $(A_1B_1C_1)$. The triangles of the second net evidently correspond to those of the first since parallel lines are transformed into parallel lines.

Consider any area Δ and the corresponding area Δ_1. Each area is made up of the same number of whole triangles and parts of triangles. Hence we have

$$\frac{\Delta}{\Delta_1}=\frac{nA+e}{nA_1+e_1} \qquad (18)$$

where A is the area of the triangle ABC and A_1 that of the triangle $A_1B_1C_1$, and where e is the sum of all the pieces of triangles within Δ and e_1 the sum of all the pieces of triangles within Δ_1. We can make the quantities e and e_1 as small as we please by taking the triangle ABC and its corresponding triangle $A_1B_1C_1$ sufficiently small. Hence we have in the limit

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*A. Emch, Annals of Mathematics, vol. 10, pp. 2–4.

$$\frac{\triangle}{\triangle} = \frac{A}{A_1} = \text{const.} \tag{18a}$$

THEOREM 23. A collineation of the group $G_6(l_\infty)$ alters all areas by a constant ratio R.

355. *Analytic Proof of Same Property.* Let (x, y), (x', y') and (x'', y'') be the vertices of any triangle A. The corresponding vertices of A_1 are given by

$$\begin{aligned} x_1 &= ax + by + c, & y_1 &= a'x + b'y + c'; \\ x_1' &= ax' + by' + c, & y_1' &= a'x' + b'y' + c'; \\ x_1'' &= ax'' + by'' + c, & y_1'' &= a'x'' + b'y'' + c'. \end{aligned} \tag{19}$$

Forming the determinant which is twice the area of A_1 we have

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_1' & y_1' & 1 \\ x_1'' & y_1'' & 1 \end{vmatrix} = \begin{vmatrix} ax + by + c & a'x + b'y + c' & 1 \\ ax' + by' + c & a'x' + b'y' + c' & 1 \\ ax'' + by'' + c & a'x'' + b'y'' + c' & 1 \end{vmatrix} = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{vmatrix} \cdot \begin{vmatrix} x & y & 1 \\ x' & y' & 1 \\ x'' & y'' & 1 \end{vmatrix} \tag{20}$$

$$\therefore A_1 = DA.$$

The ratios of the two areas is therefore D which is the determinant of the collineation T.

356. *Types of Collineations in $G_6(l_\infty)$.* It may be shown that the group $G_6(l_\infty)$ contains all five types of plane collineations. When a collineation T of type I occurs in this group, one side of the invariant triangle of T must coincide with the line at infinity, l_∞. A collineation T' of type II may occur in $G_6(l_\infty)$ in two different ways; the line l or the line l' may coincide with l_∞. Collineations of type III occur in $G_6(l_\infty)$ when the invariant line of T'' coincides with l_∞. A perspective collineation S of type IV may occur in $G_6(l_\infty)$ in either of two ways; the axis l of the collineation may coincide with l_∞, or the vertex A may be on the line at infinity and the axis pass through finite space. A collineation S' of type V may occur in either of two ways in $G_6(l_\infty)$; the axis may coincide with l_∞, or the vertex A may be on l_∞ and some line of the pencil through A coincide with l_∞.

357. *Value of R, the Ratio of Areas.* Since every collineation in $G_6(l_\infty)$ alters all areas in a constant ratio R, it is important to find the value R in terms of the natural parameters of the collineation. Each type of collineation requires separate treatment.

Type I. Let us consider a collineation T and its invariant triangle having the side $A'A''$ at infinity. Let PQ, Fig. 32, be a tangent to a path-curve C and let it be transformed into another tangent, P_1Q_1, to the same path-curve. The area APQ is transformed into AP_1Q_1, and we wish to find the ratio of these areas. The cross-ratio along the side AA' is $R = (A \infty P_1 P) = \frac{AP_1}{AP}$. The cross-ratio along the side AA'' is $R = (A \infty Q_1 Q) = \frac{AQ_1}{AQ}$. The ratios of the two areas are given as follows:

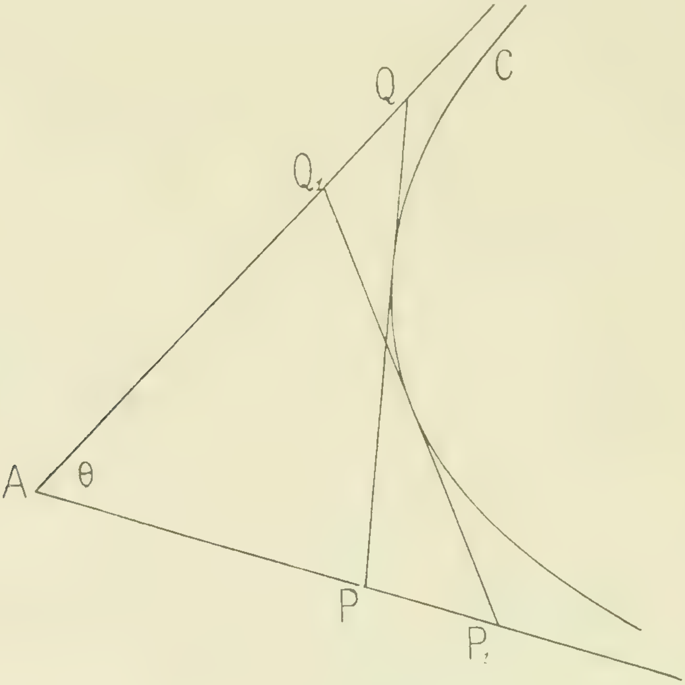

FIG. 32.

$$R \equiv \frac{\text{Area of } AP_1Q_1}{\text{Area of } APQ} = \frac{AP_1 \,.\, AQ_1 \sin\theta}{AP \,.\, AQ \sin\theta} = kk'.$$

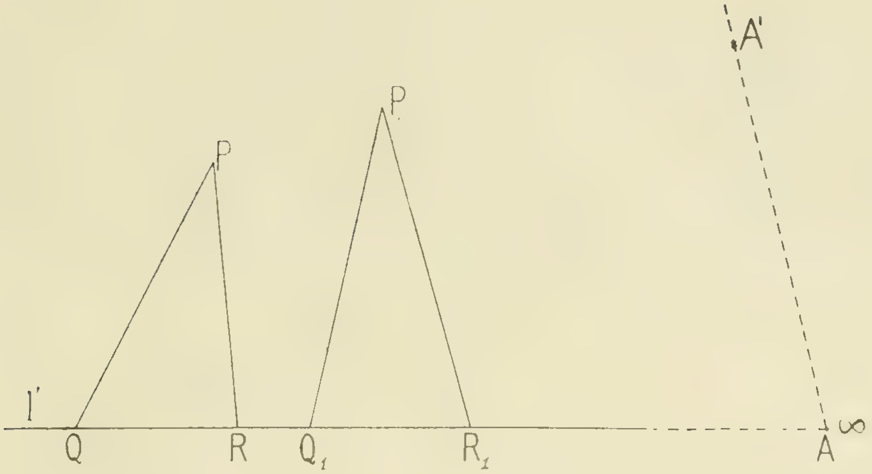

FIG. 33.

358. *Type II.* In the first case let the line AA', Fig. 33, be the line at infinity and let the triangle PQR be transformed by T' into $P_1Q_1R_1$, Q and R being two points on Al'. The one-dimensional transformation along l' is an Euclidian translation (Art. 108), and hence $Q_1R_1 = QR$. The cross-ratio of the transformations of the pencil through A is $A(l'A'P_1P) = k = (0\, \infty\, p_1 p) = \frac{p_1}{p}$ where p and p_1 are the perpendiculars from P and P_1 to l. Hence $R = \frac{\triangle P_1Q_1R_1}{\triangle PQR} = \frac{p_1}{p} = k$.

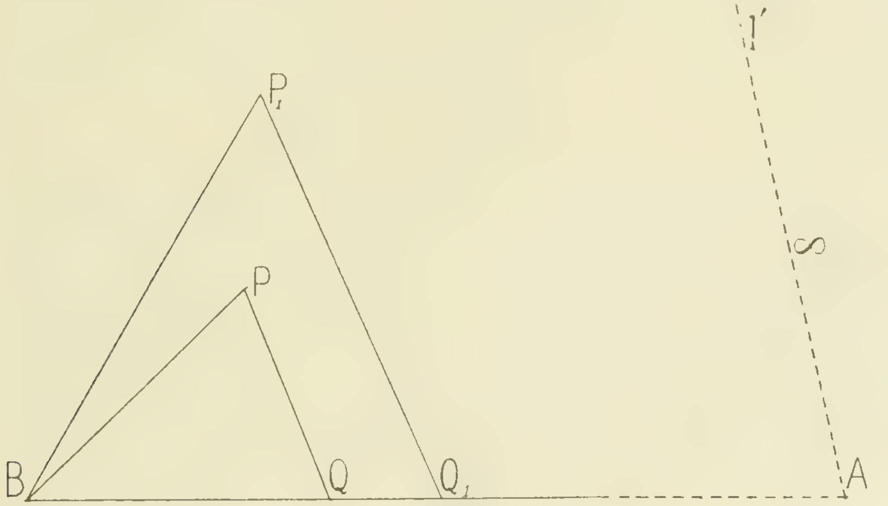

FIG. 34.

In this case R the ratio of areas is equal to k the cross-ratio of the transformation of the pencil through A.

In the second case let the line Al', Fig. 34, be the line at infinity and let the triangle BPQ be transformed into BP_1Q_1. The cross-ratio along BA is $k = (B \infty Q_1 Q) = \frac{BQ_1}{BQ}$. The cross-ratio of the pencil through A is also $k = A(\theta \infty p_1 p) = \frac{p_1}{p}$. Hence we have $R = \frac{\triangle BP_1Q_1}{\triangle BPQ} = \frac{BQ_1 . p_1}{BQ . p} = k^2$.

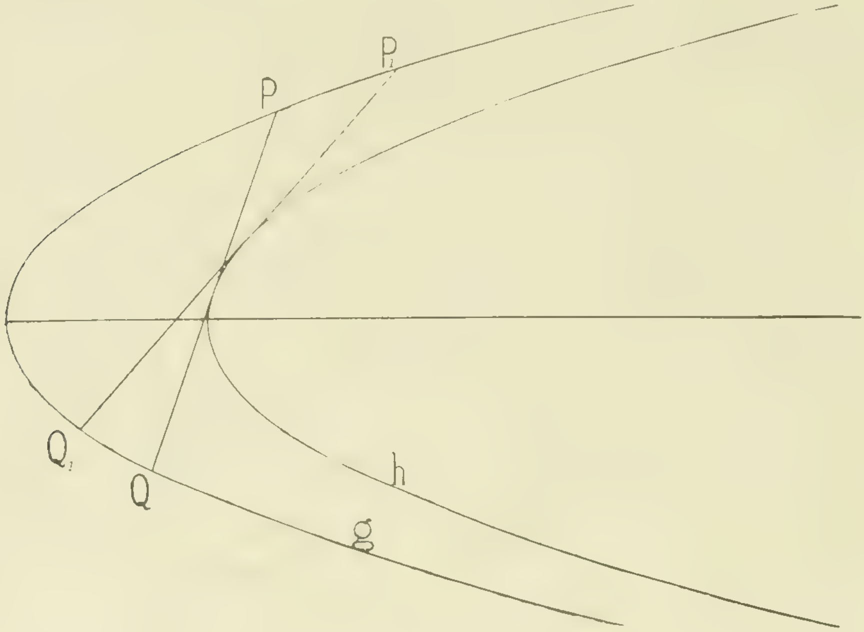

FIG. 35.

359. *Type III.* In this case the path-curves of the one-parameter group $G_1''(AlS)$ to which T'' belongs are parabolas similar and similarly placed; *i. e.*, coaxial parabolas having equal *latera recta*. Consider the area of the segments of the parabola g cut off by tangents to h, Fig. 35. If T'' transforms PQ into P_1Q_1, then the area cut off from g by PQ is transformed into the area cut off by P_1Q_1. But these areas are known to be equal, (Salmon, 396); hence, $R = 1$.

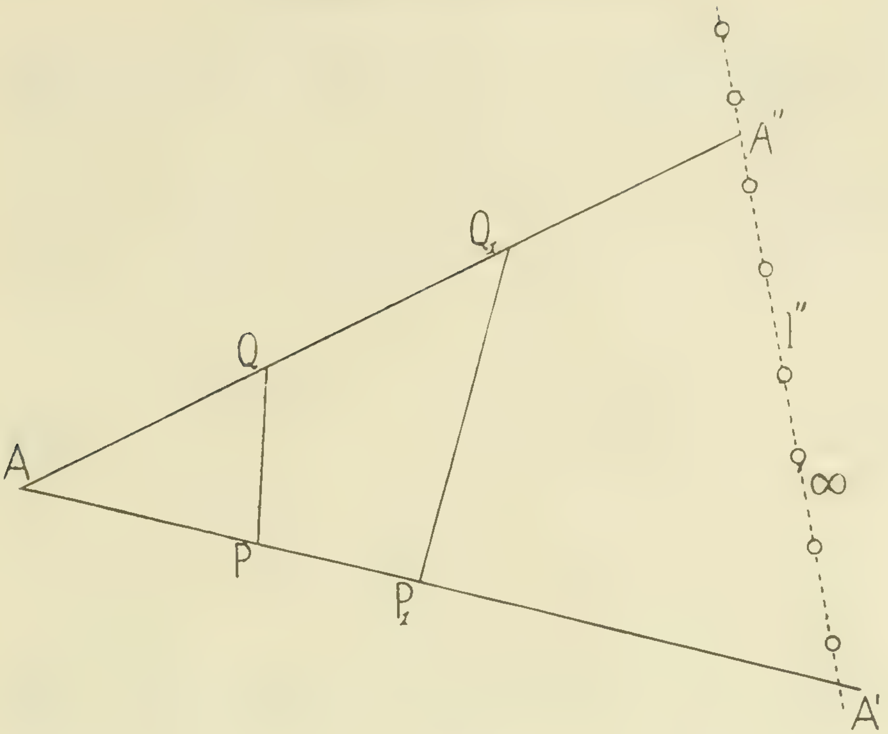

FIG. 36.

360. *Type IV.* In the first place let l'' be the line at infinity, and let A be in finite space, Fig. 36. The triangle APQ is transformed into AP_1Q_1. Along AA'', $k=\frac{AQ_1}{AQ}$; along AA', $k=\frac{OP_1}{OP}$. Hence we have

$$R=\frac{\triangle AP_1Q_1}{\triangle APQ}=\frac{AP_1.AQ_1 \sin\theta}{AP.AQ \sin\theta}=k^2.$$

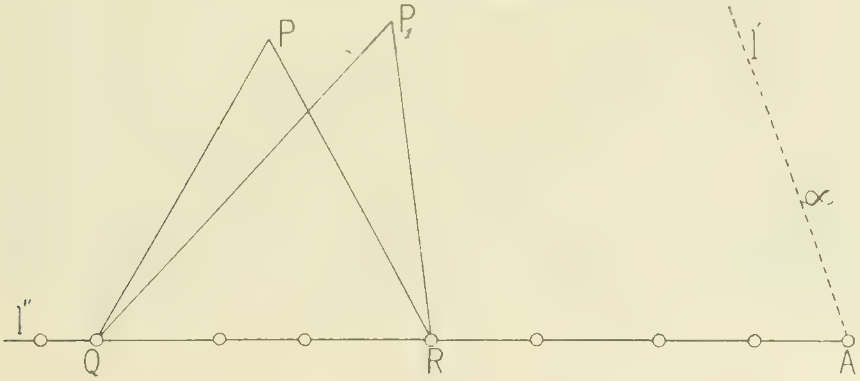

FIG. 37.

In the second case, let Al', Fig. 37, be the line at infinity, and let the triangle PQR be transformed into P_1QR, Q and R being two invariant points on l''. The cross-ratio of the pencil through A is $k = \frac{p_1}{p}$ where p_1 and p are perpendiculars from P_1 and P on l''. Hence we have $R = \frac{\triangle P_1QR}{\triangle PQR} = \frac{p_1}{p} = k$.

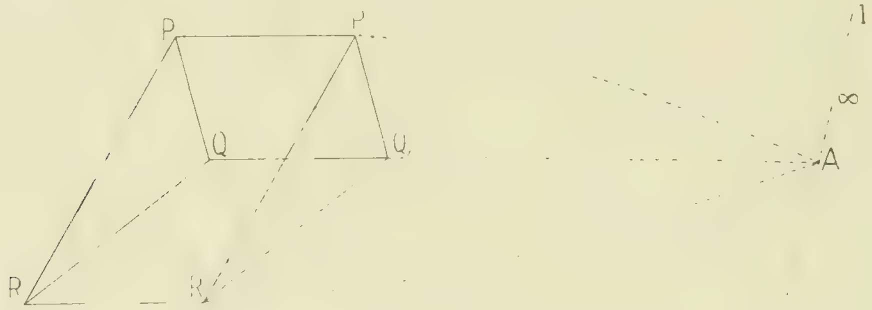

FIG. 38.

361. *Type V.* In the first case, when the line l is the line at infinity, the triangle PQR is transformed into $P_1Q_1R_1$. But PP_1, QQ_1 and RR_1 are all parallel, and $PP_1 = QQ_1 = RR_1$ for the one-dimensional transformations along PA, QA, RA are all translations of equal length. Hence $R = \frac{\triangle P_1Q_1R_1}{\triangle PQR} = 1$ and all areas are unaltered.

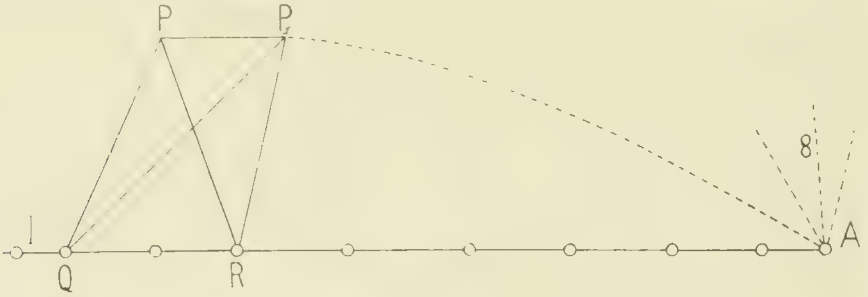

FIG. 39.

In the second case, Fig. 39, when the point A is at infinity and l passes through finite space, the triangle PQR is transformed into P_1QR where Q and R are two invariant points on

l. But the line PP_1 passes through A and is parallel to l. Hence the triangles PQR and P_1QR are equal in area.

$\therefore R = 1$, *i. e.*, areas are unaltered by S.

THEOREM 24. The value of R, the ratio of areas for a collineation in the group $G_6(l_\infty)$, is as follows: For type I, $R = kk'$; for type II, $R = k$ in the first case, and $R = k^2$ in the second case; for type III, $R = 1$; for type IV, $R = k^2$ in the first case, and $R = k$ in the second case; for type V, $R = 1$ in both cases.

362. *Five-parameter Subgroups of* $G_6(l_\infty)$. Every six-parameter group $G_6(l)$ leaving invariant any line l of the plane, breaks up into ∞^1 five-parameter groups of the variety $G_5(Al)$ leaving a lineal element Al invariant; it also contains, Art. 209 one five-parameter subgroup of another type, viz.: $G_5(l)_{r=-1}$, composed of one-parameter subgroups whose path-curves are conics having l for common chord of contact.

Having proved the existence of such a unique subgroup for every line l, we wish to study in particular this subgroup when l is the line at infinity. It will be shown that this special subgroup of $G_6(l_\infty)$ transforms areas into equal areas. It is therefore called the group of *Invariant Areas.*

363. *The Special Linear Group.* The general linear group is given by the equation

$$x_1 = ax + by + c, \quad y_1 = a'x + b'y + c'.$$

The determinant of this collineation is $D = \begin{vmatrix} a & b & c \\ a' & b' & c' \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} a & b \\ a' & b' \end{vmatrix}$.

A second collineation of the same group has the determinant $D_1 = \begin{vmatrix} a_1 & b_1 \\ a_1' & b_1' \end{vmatrix}$. The value of the determinant of the resultant of these two collineations is, by Art. 172, $D_2 = DD_1$. If now D and D_1 are both equal to unity then D_2 is also equal to unity. The resultant of any two collineations of the general linear group whose determinants are both unity is a collineation of the same group with determinant also unity. Hence all collineations of the general linear group whose determinants are equal to unity form a subgroup of the general linear group.

This subgroup is called the Special Linear Group and is given by the equations

$$x_1 = ax + by + c, \quad y = a'x + b'y + c',$$

with the condition $\begin{vmatrix} a & b \\ a' & b' \end{vmatrix} = 1$.

364. *Areas are Transformed Into Equal Areas.* It was shown in Art. 355 that a collineation of the general linear group alters all areas of the plane in the same ratio. Thus it was shown that $\Delta_1 = D\Delta$, when Δ is the original area, Δ_1 the transformed area, and D the determinant of the collineation. In the special linear group $D = 1$; hence $\Delta_1 = \Delta$, *i. e.*, every area is transformed into an equal area.

THEOREM 25. The Special Linear Group transforms every area into an equal area; *i. e.*, it is the group of Invariant Areas $G_5(l_\infty)\Delta$.

365. *Collineations of Type I in $G_5(l_\infty)\Delta$.* It was shown in Art. 357 that for a collineation of type I in $G_6(l_\infty)$ the constant ratio of areas is $R = kk' = k^{1+r}$. If this ratio is equal to unity, then we must have $r = -1$. But if $r = -1$ then the path-curves of the one-parameter group $G_1(ABC)_{r=-1}$ are conics, Art. 254, having AA' and AA'' for common tangents and $A'A''$ for chord of contact. The line $A'A''$ is now the line at infinity and hence the conics having double contact at A' and A'' are concentric conics having the same asymptotes. When A' and A'' are real points, the path-curves are concentric hyperbolas having the same asymptotes. When A' and A'' are conjugate imaginary points the path-curves of a one-parameter group are similar and concentric ellipses.

In the case of a hyperbolic one-parameter subgroup of $G_5(l_\infty)$ (when A' and A'' are real), we can readily see that the areas are transformed into equal areas, for all tangents to a hyperbola, Fig. 40, form with the asymptotes triangles of equal areas. Likewise all segments of one hyperbola cut off by tangents to a similar and concentric hyperbola have equal areas (Salmon 396). In the case of an elliptic subgroup, where the path-curves are similar and concentric ellipses, areas are evidently transformed into equal areas.

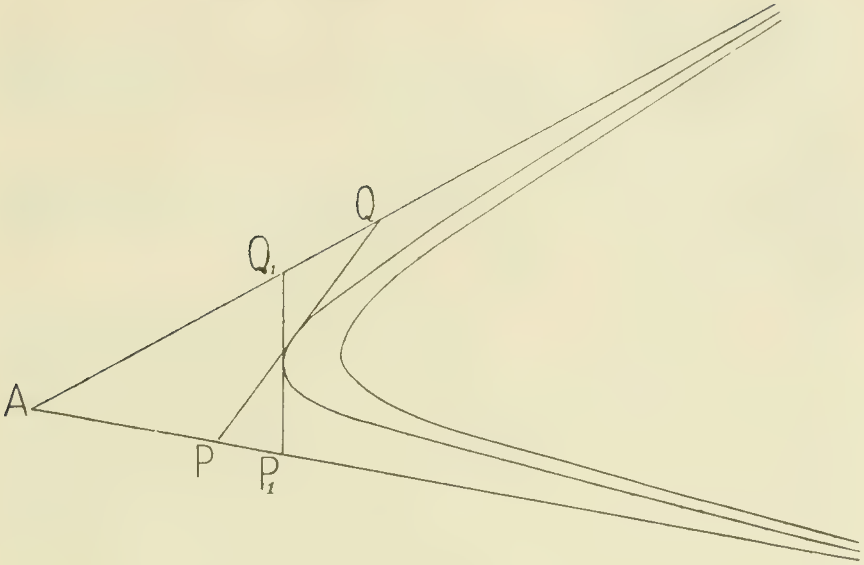

FIG. 40.

Since there are ∞^4 triangles in the plane having the line at infinity for one side, it follows that the group $G_5(l_\infty)_\triangle$ contains ∞^4 one-parameter subgroups of type I, all of whose path-curves are conics having double contact on the line at infinity.

366. *Collineations of Type III in* $G_5(l_\infty)_\triangle$. It was proved in Art. 359 that a collineation of type III in $G_6(l_\infty)$ has the constant ratio of areas $R = 1$. Therefore all such collineations in $G_6(l_\infty)$ transform areas into equal areas and consequently belong to the subgroup $G_5(l_\infty)_\triangle$.

There are ∞^4 collineations of type III in $G_5(l_\infty)_\triangle$, for there are ∞^1 linear elements on l and each lineal element is the invariant figure of ∞^3 collineations T''. These ∞^4 collineations T'' fall into ∞^3 one-parameter subgroups $G_1''(Al_\infty)$, ∞^2 for each point on l_∞. The ∞^4 parabolas of the plane can be arranged in ∞^3 pencils of similar and coaxial parabolas; each of the pencils constitutes the path-curves of one of the ∞^3 one-parameter subgroups $G_1''(Al_\infty)$.

367. *Collineations of Type V in* $G_5(l_\infty)_\triangle$. All collineations of type V in $G_6(l_\infty)$ transform areas into equal areas,

for it was shown in Art. 361 that for this type $R = 1$. All collineations of type V leaving the line at infinity invariant form a two-parameter group $H_2'(l_\infty)$, Art. 201, and this is therefore a subgroup of $G_5(l_\infty)_\triangle$.

THEOREM 26. The group of invariant areas $G_5(l_\infty)_\triangle$ consists of collineations of types I, III, and V; it contains ∞^4 one-parameter subgroups $G_1(ABC)_{r=-1}$; ∞^3 one-parameter subgroups $G_1''(AlS_\infty)$; and ∞^1 one-parameter subgroups $H_1'(Al_\infty)$.

368. *The Group of Similarity*, $G_4(\omega\omega')$. We shall now take up the detailed study of the real four-parameter group $G_4(\omega\omega')$ whose invariant figure is the line at infinity and the two circular points at infinity ω and ω'. The group $G_4(\omega\omega')$ evidently contains ∞^2 two-parameter subgroups $G_2(A\omega\omega')$, where A is any point in the plane forming with ω and ω' the triangle $(A\omega\omega')$. Each of these two-parameter groups $G_2(A\omega\omega')$ contains ∞^1 one-parameter subgroups $G_1(A\omega\omega')_r$.

Since two of the vertices of the invariant triangle $(A\omega\omega')$ are conjugate imaginary points it follows that all the collineations of type I in $G_4(\omega\omega')$ are elliptic. There can be no real collineations of type II in $G_4(\omega\omega')$ because for such collineations the invariant points A and A' are always real points and hence can not be made to coincide with ω and ω'. There can be no real collineations of type III in $G_4(\omega\omega')$ for similar reasons.

$G_4(\omega\omega')$ contains real collineations of types IV and V; for if the line of invariant points of a collineation of type IV or V be the line at infinity, the imaginary points ω and ω' as well as all real points on the line l_∞ are invariant.

369. *Angular Magnitudes are Invariant.* A collineation $T(\omega\omega')$ which transforms any point P into P_1 necessarily transforms the lines joining P to ω and ω' into the lines joining P_1 to ω and ω'. Thus isotropic* lines are transformed into isotropic lines; the system of isotropic lines in the plane is invariant under the group $G_4(\omega\omega')$.

Two lines p and p' meeting in P are transformed by $T(\omega\omega')$

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* C. A. Scott, Modern Analytic Geometry, Art. 113.

into p_1 and p_1' meeting in P_1. Since cross-ratios are unaltered by a collineation, the cross-ratio of the pencil $P(pp'\omega\omega')$ is equal to that of $P_1(p_1p_1'\omega\omega')$. But the cross-ratio of the pencil formed by two intersecting lines p and p' and the isotropic lines through the point of intersection measures the angle between p and p'.* Hence the angle between p and p' is equal to the angle between the two corresponding lines p_1 and p_1'. This is true for all points of the plane. Therefore all angles in the plane are transformed into equal angles; in other words, all angular magnitudes are invariant for the group $G_4(\omega\omega')$.

Since all angular magnitudes are unaltered by a collineation $T(\omega\omega')$, it follows that any figure is transformed into a similar figure. Shape is conserved but size is not necessarily conserved. The group $G_4(\omega\omega')$ is therefore called the Group of Similarity. The group $G_4(\omega\omega')$ is a subgroup of $G_6(l_\infty)$; hence every collineation in $G_4(\omega\omega')$ alters all areas by a constant ratio R. But if angular magnitudes are unaltered and areas are altered by a constant ratio R, it follows that all linear magnitudes are also altered by a constant ratio R' which is given by $R' = \sqrt{R}$.

THEOREM 27. Every collineation of the group $G_4(\omega\omega')$ leaving invariant the line at infinity and the two circular points, transforms angles into equal angles and every figure into a similar figure and alters every linear magnitude by a constant ratio.

370. *The Path-curves of the Group* $G_1(A\omega\omega')_r$. The path-curves of a one-parameter group $G_1(A\omega\omega')_r$ require special attention because of their remarkable form. We proved in Art. 252 that the path-curves of the one-parameter group $G_1(ABC)_r$ have the property that the cross-ratio of the pencil formed by a tangent to a path-curve at P, and the three lines from the point of contact to the invariant points is constant and equal to r for all points of the plane. Since $P\omega$ and $P\omega'$ are the isotropic lines through P, it follows that the angle between AP and the tangent to the path-curve at P is constant for all points of the plane. The only plane curve for which

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* C. A. Scott, Modern Analytic Geometry, Art. 273.

the angle between the radius vector and the tangent at its extremity is constant is the logarithmic spiral. Hence the path-curves of the one-parameter group $G_1(A\omega\omega')_r$ are logarithmic spirals about the point A_1; these spirals cut all the lines through A at a constant angle.

It may also be shown from the equations to the path-curves of $G_1(AA'A'')_r$ that, when A' and A'' are the circular points ω and ω', these path-curves become spirals. The equation of the path-curves of $G_1(AA'A'')_r$ is, Art. (251),

$$x^r = Cyz^{r-1}.$$

Putting $z = 1$ the side $A'A''$ of the invariant triangle is shifted to infinity. To make A' and A'' the circular points at infinity, we write $x - iy$ for x and $x + iy$ for y (or *vice versa*). The above equation then becomes

$$x + iy = C(x - iy)^r.$$

Setting $r = a + ib$ we have

$$x + iy = Ce^{-i\theta(a+ib)} = Ce^{b\theta}e^{-ai\theta}.$$

Also by changing the sign of i

$$x - iy = Ce^{b\theta}e^{ai\theta}.$$

Multiplying we get

$$x^2 + y^2 = C^2e^{2b\theta}.$$

Setting $x^2 + y^2 = \delta^2$ we have

$$\delta = Ce^{b\theta},$$

which is the polar equation of a family of logarithmic spirals.

THEOREM 28. The path-curves of a one-parameter group $G_1(\omega\omega')$ whose invariant triangle has two vertices at the circular points at infinity are a family of logarithmic spirals about A.

371. *Collineations of Types IV and V in* $G_4(\omega\omega')$. The group $G_4(\omega\omega')$ contains the three-parameter group $H_3(l_\infty)$; for the ∞^3 perspective collineations $S(l_\infty)$ form a group and in the invariant figure of this group is every point on the line at infinity, including therefore the two circular points ω and ω'. Within this group $H_3(l_\infty)$ is the two-parameter group

$H_2'(l_\infty)$ of elations; this group of elations is therefore a subgroup of $G_4(\omega\omega')$.

372. *One-parameter Groups of Dilations.* The path-curves of a group $H_1(A, l_\infty)$ are the pencil of lines through A. Since the characteristic cross-ratio of a collineation S of the group $H_1(A, l_\infty)$ is constant along all lines through A, it follows that S produces a dilation of the whole plane, A being the center of dilation. All linear magnitudes are altered in the constant ratio k and all areas in the constant ratio k^2. For negative values of k any figure F and its corresponding figure F_1 are situated symmetrically on opposite sides of A.

373. *The Mixed Group* $mG_4(\omega\omega')$. Having pointed out all varieties of collineations that leave the circular points ω and ω' separately invariant, we proceed to consider those collineations which interchange ω and ω'. The mixed group $mG_4(\omega\omega')$ is only a special case of $emG_4(PQ)$, which group was investigated in § 3 of this chapter. We thus see that $emG_4(\omega\omega')$ contains ∞^4 collineations of type I, ∞^3 collineations of type II, and ∞^2 involutoric collineations of type IV.

Let $AA'A''$ be the invariant triangle of a hyperbolic collineation of type I which interchanges ω and ω'. Since A' and A'' separate ω and ω' harmonically, we see that the angle $A'AA''$ is a right angle. Hence such a collineation leaves invariant a finite point A and two lines through A at right angles to one another. Every angle in the plane is unaltered in magnitude but reversed in sense by such a collineation, while all areas are altered by a constant ratio, which ratio R is always negative. The effect of a collineation of this kind is to transform every plane figure into a similar but noncongruent figure. With AA' and AA'' for axes the collineation T is given by $x_1 = kx$, $y_1 = ky$ with the condition $k + k' = 0$.

Let $AA'l'$ be the invariant figure of a collineation T' of type II which interchanges ω and ω'. Since the points A and A' separate ω and ω' harmonically, we see that lines perpendicular to l', the finite invariant line of T', are transformed into lines also perpendicular to l'. A line parallel to l' is transformed into another line also parallel to l', but at an equal distance from l on the other side. Angles are unaltered in magnitude but reversed in sense, and linear magnitudes are reversed in sense but unaltered in length. The effect of such a collineation is to revolve the whole plane through two right angles about the line l' as an axis, and then to slide the whole plane along l'. With the origin on l the collineation T' is given by $x_1 = -x$, and $y_1 = y + t$.

Let $t = 0$ in the above collineation T' of type II; T' then reduces to an involutoric collineation of type IV. The effect of such a collineation is to revolve the whole plane through two right angles about l' as an axis.

THEOREM 29. The mixed group $em\,G_4(\omega\omega')$ contains, besides the continuous group $e\,G_4(\omega\omega')$, ∞^4 hyperbolic collineations of type I, ∞^3 collineations of type II, and ∞^2 involutoric collineations of type IV. All collineations interchanging ω and ω' transform plane figures into similar but non-congruent plane figures.

374. *The Group of Euclidian Motions.* The group of Euclidian motions in the plane is a subgroup of the group of similarity $G_4(\omega\omega')$. Every collineation in $G_4(\omega\omega')$ transforms all plane figures into similar and congruent figures and alters all linear magnitudes in a constant ratio R; shape is an invariant of the group $G_4(\omega\omega')$, but size is not invariant.

All collineations in $G_4(\omega\omega')$ for which R is unity form a subgroup of the group of similarity; all such transformations are common to the two groups $G_5(l_\infty)_{r=-1}$ and $G_4(\omega\omega')$ and form a subgroup of each. All collineations of this subgroup conserve size as well as shape. Every plane figure is transformed into an equal and congruent plane figure. But such a collineation is evidently brought about by a rigid motion of the whole plane into itself. Hence the group of collineations of the plane which conserves the size, shape and congruity of every plane figure is the group of all Euclidian motions in the plane.

375. *The Group of Motions is $G_3(\omega\omega')_{r=-1}$.* It was shown in Art. 370 that the path-curves of the one-parameter group

$G_1(A\omega\omega')_r$ are logarithmic spirals, $r = Ce^{b\theta}$, where b is the cotangent of the constant angle ψ between the radius vector and the tangent to the curve. When $b = 0$, $\cot \psi = 0$ and $\psi = \frac{\pi}{2}$. In this case the spirals degenerate into concentric circles about the point A. R, the ratio of linear magnitude, in this case is unity.

The path-curves of this one-parameter group are concentric circles; *i. e.*, they are conics having double contact at ω and ω'. The group is therefore evidently $G_3(A\omega\omega')_{r=-1}$ or $G_3(EM)$. There are ∞^2 such one-parameter groups in the plane, one for each point A. When the point A falls on the line at infinity, the collineation is no longer of type I but degenerates, as we shall see, into a collineation of type V.

376. *Collineations of Type V in $G_3'(EM)$.* The group of similarity was found to contain collineations of types I, IV and V (Art. 368). A collineation belonging to the group of similarity is a motion when the ratio of expansion is unity. We saw in the last article how the ratio of expansion might be unity for a collineation of type I.

The ratio of expansion of a collineation of type IV in the group of similarity is $R = k$; so that if $R = 1$, then also $k = 1$. But $k = 1$ is the identical transformation in the group $H_1(Ol_\infty)$; hence we see that the group of motions contains no collineation of type IV.

The ratio of expansion for a collineation of type V in the group of similarity is $R = 1$; hence all collineations of type V in the group of similarity are also to be found in the group of motions.

The two-parameter group of type V, $H_2'(l_\infty)$, is therefore a subgroup of the group of motions. This group $H_2'(l_\infty)$ is common to $G_6(l_\infty)$, $G_5(l_\infty)_r$, $G_4(\omega\omega')$ and the group of motions, $G_3(\omega\omega')_{r=-1}$.

377. *Motion is Either a Rotation or a Translation.* The group of motions contains only collineations of type I and type V. These must be examined separately. The path-curves of a one-parameter group of motions of type I are con-

centric circles about the point A. The collineations of this group are therefore rotations of the whole plane about the point A. It is evident that all rotations of the plane about a fixed point A form a one-parameter group. A rotation about A through an angle θ combined with another about the same point through an angle θ_1 results in a rotation about the same point through an angle $\theta_2 = \theta + \theta_1$. The characteristic cross-ratios of the two component collineations are respectively $e^{i\theta}$ and $e^{i\theta_1}$; the cross-ratio of the resultant is $e^{i\theta_2} = e^{i\theta} \cdot e^{i\theta_1} = e^{i(\theta+\theta_1)}$.

The path-curves of a one-parameter group of motions of type V are straight lines meeting at A, a point on the line at infinity; *i. e.*, they are parallel lines all in the direction of A. The collineations of this one-parameter group are therefore translations of the whole plane in the direction of A. It is evident that all translations of the plane in a given direction form a group; a translation in a given direction through a distance t combined with another translation in the same direction through a distance t_1, results in a translation in the same direction through the distance $t_2 = t + t_1$.

THEOREM 30. The group of Euclidian motions $G_3(\omega\omega')_{r=-1}$ contains only collineations of types I and V; the former are Rotations of the whole plane about a point, the latter are Translations of the whole plane in a fixed direction.

378. *Subgroups of the Group of Motions.* The group of motions contains ∞^3 distinct rotations and ∞^2 distinct translations. All the rotations about a point form a group of one-parameter and there are ∞^2 such points in the plane; hence all the rotations of the plane naturally fall into ∞^2 one-parameter subgroups of rotations, one subgroup for each finite point in the plane. These one-parameter groups of rotations do not combine to form two-parameter groups of rotations, for the resultant of two rotations is not always a rotation but is sometimes a translation.

The ∞^2 translations of the plane form a two-parameter subgroup of the group of motions. This is the group $H_2'(l,\ )$.

It is evident geometrically that the resultant of any two translations is a translation and hence that all translations of the plane form a group. A translation of the whole plane from O to P followed by a translation from P to P_1 is equivalent to a translation of the whole plane from O to P_1.

The two-parameter group of translations $H_2'(l_\infty)$ breaks up into ∞^1 one-parameter subgroups of translations, one for each point on the line at infinity, *i. e.*, one for each direction in the plane.

379. *The Mixed Group* $mG_3(EM)$. We found in Art. 373 that the transformations in $mG_4(\omega\omega')$ which interchange ω and ω' are of types I, II, and IV. All of these transformations which change areas into equal areas belong to the mixed group of Euclidian motions. These we now proceed to examine.

The two characteristic cross-ratios k and k' of a hyperbolic transformation T of type I which interchange ω and ω' satisfy the condition $k+k'=0$; Art. 373. If T leaves all areas invariant in magnitude but reversed in sense, k and k' must satisfy the relation $kk'=-1$; Art. 361. From these two conditions we have $\begin{cases} k=1 \\ k'=-1 \end{cases}$ or $\begin{matrix} k=-1 \\ k'=1 \end{matrix}$; in either case we see that the collineation is no longer of type I but of type IV, *i. e.*, it is a perspective collineation. Thus the mixed group $mG_3(EM)$ contains no hyperbolic collineations of type I.

All collineations of type II in $emG_4(\omega\omega')$ interchange ω and ω' and transform all plane areas into equal but non-congruent plane areas. Consequently the ∞^1 collineations of type II interchanging ω and ω' belong to the mixed group $mG_3(EM)$. The effect of such a collineation, Art. 373, is to revolve the whole plane through an angle of 180° about some line of the plane as an axis and then to slide the whole plane along the axis.

All transformations of type IV in $mG_4(\omega\omega')$ evidently transform areas into equal but non-congruent areas and hence belong to the group $mG_3(EM)$. The effect of such a transformation is to revolve the whole plane through 180° about the line of invariant points as an axis.

THEOREM 31. The mixed group of Euclidian motions contains, besides the continuous group $eG_3(EM)$, ∞^3 collineations of type II and ∞^2 collineations of type IV; these collineations interchange ω and ω' and transform all plane areas into equal but non-congruent plane areas.

Exercises on Chapter 4.

1. Verify synthetically and analytically the structural formulas of all perspective groups given in Art. 315.

2. Show that the group $G_3''(AlN)$ contains $H_1'(Al)$ as a subgroup.

3. Verify synthetically and analytically the structural formulas of all groups of type II, first class, as given in Art. 317.

4. Verify by both methods the structural formulas of all groups of type II, second class, as given in Art. 318.

5. Verify the structure of all groups of type I, first class, as given in Art. 319.

6. Verify by both methods the structure of all groups of type I, second class, as given in Art. 320.

7. Verify by both methods the structure of all groups of type I, third class, as given in Art. 321.

8. Show that group $G_3'(Al)_a$ contains singular transformations of both types III and V; and the group $G_3(AA')_a$ contains singular transformations of type III.

9. Show that groups $G_3(AA')_r$, $G_3(ll')_r$, $G_4(Al)_r$ (when r is rational), each contain singular transformations of type II.

10. Show that the group $G_5(A)_{r=-1}$ contains ∞^4 singular transformations of type II.

11. Show that the singular transformations of type II in

$G_3(AlS)$ are the resultants of a system of involutoric perspective collineations with the elations of the subgroup $H_1'(Al)$.

12. Show that the system of collineations selected from the group $G_2(AA'A'')$, so as to satisfy the condition $k+k'=0$, form a continuous system but not a continuous group.

13. Show that the system of parabolic transformations within $G_1'(A)$, which correspond on the Argand diagram to all points on a straight line through the origin, has both group properties and hence forms a subgroup of $G_1'(A)$.

14. Show that there is one such subgroup of $G_1'(A)$ for each line through the origin; hence show that the real group $pG_1(A)$ is a subgroup of $G_1'(A)$, when A is real.

15. Show that the system of loxodromic transformations within $G_1(AA')$, which correspond on the Argand diagram to all points on a logarithmic spiral $r=e^{(c+i)\theta}$ about the origin and through the unit point, has both group properties and hence forms a subgroup of $G_1(AA')$.

16. Show that $G_1(AA')$ contains one such subgroup for each value of c; hence show that $cG_1(AA')$ is a subgroup of $G_1(AA')$ when A and A' are conjugate imaginary points; show also that $hG_1(AA')$ is a subgroup of $G_1(AA')$ when A and A' are a pair of real points.

17. Show that each of the plane collineation groups, $H_1'(Al)$, $G_1''(AlS)$, and $G_1(AA'l)_a$, in which the parameter t assumes in turn all complex values, breaks up into ∞^1 continuous subgroups, one for each value of θ in $t=\pm re^{i\theta}$.

18. Show that each of the plane collineation groups, $H'(Al)$ and $G_1(AA'A'')_r$, in which k assumes in turn all complex values, breaks up into ∞^1 continuous subgroups, one for each value of c in $k=e^{(c+i)\theta}$.

19. Show that those hyperbolic transformations in $hG_1(AA')$, for which k is negative and which therefore can not be generated from either real infinitesimal transforma-

tion in $hG_1(AA')$, can be generated from complex infinitesimal transformations in $G_1(AA')$.

20. Show that a real collineation $hT(AA'A'')$ of type I, for which k and k' are both positive, belongs to one and only one one-parameter subgroup of $hG_2(AA'A'')$.

21. Find the determinant of the normal form of a collineation of type I in $G_6(l_\infty)$, and hence show analytically that R, the ratio of areas, is kk'.

22. Find in a similar manner the value of R for collineations of types II, III, IV and V, in $G_6(l_\infty)$.

23. Show analytically that the group of invariant areas is composed of collineations of types I, III and V.

24. Deduce from the equations of the normal form of type I the following equations of the Group of Similarity, $G_4(\omega\omega')$:

$$x_1 = \tfrac{1}{2}(k+k')x + \tfrac{i}{2}(k-k')y + A - \tfrac{A}{2}(k+k') + \tfrac{B}{2i}(k-k'),$$
$$y_1 = -\tfrac{i}{2}(k-k')x + \tfrac{1}{2}(k+k')y + B - \tfrac{A}{2i}(k-k') - \tfrac{B}{2}(k+k').$$

25. Deduce from the equations of problem 24 the equations of the group $H_2(l_\infty)$ and show that this group is a subgroup of $G_4(\omega\omega')$.

26. Show that $H_2'(l_\infty)$ is also a subgroup of $G_4(\omega\omega')$ and deduce its equations from those of problem 24.

27. Deduce from the equations of problem 24 the following equations of a rotation about the point (AB) through an angle θ:

$$x_1 = x\cos\theta - y\sin\theta + A - A\cos\theta + B\sin\theta,$$
$$y_1 = x\sin\theta + y\cos\theta + B - B\cos\theta - A\sin\theta.$$

28. Prove that the resultant of two rotations, T and T_1, through angles θ and θ_1 respectively, about points (AB) and (A_1B_1) respectively, is a rotation about a third point, (A_2B_2), through an angle $\theta_2 = \theta + \theta_1$, and find the coordinates (A_2B_2) of the new centre of rotation.

29. Prove that the resultant of two rotations through equal angles in different directions about different points is a translation perpendicular to the line joining the two points.

30. Prove analytically that the resultant of a rotation and a translation is a rotation.

31. Prove analytically that the resultant of two translations is a translation.

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THE PENNSYLVANIC AMPHIBIA OF THE MAZON CREEK, ILLINOIS,
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THE PENNSYLVANIC AMPHIBIA OF THE MAZON CREEK, ILLINOIS, SHALES.*

(Contribution from the Zoölogical Laboratory, No. 198.)

BY ROY L. MOODIE.

Plates 1-14.

INTRODUCTION.

THE fossil-bearing nodules from the shales along Mazon Creek, Grundy county, Illinois, have, ever since their discovery, been very prolific in excellently preserved and highly interesting forms of both plants and animals. Among the latter there have been examples of nearly all the groups which one could hope to find in a brackish or fresh-water and inland deposit. These forms range from low down in the zoölogical scale to the forms which must have stood in an ancestral relation to the reptiles. No true reptiles have been discovered in the deposits, although the discovery of such a group there would not be surprising in the least, yet of the greatest interest. A single excellently preserved skeleton of a true reptile is known from the Linton Coal of Ohio, and has been described by Cope, Williston and the writer. Two true reptiles are known also from the Carboniferous beds of Commentry, France, and examples of reptiles are known from the Rothliegenden of Saxony and from the Gaskohle of Bohemia. The lower portions of these latter deposits are undoubtedly of Upper Carboniferous age.

The fauna and flora of the Mazon Creek shales have been described by many authors. Doctor Eastman, in 1902, published a list[1] of the vertebrates known at that time from these shales. This list included twenty-five species of fishes and a single species of Amphibia, *Amphibamus grandiceps* Cope, which was all that was known of the higher vertebrates from these beds at that time. Since the publication of Eastman's essay the writer has described[2] an additional species from Mazon Creek, *Micrerpeton caudatum*. This form was shown to be an example of the order Branchiosauria. It was the first definite evidence of the occurrence of this order of Amphibia in America, or, in fact, in the Western Hemisphere.

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* Received for publication March, 1911.

The next year the writer described (Amer. Natl., XLIV, June, 1910, p. 367) and figured another branchiosaurian as *Eumicrerpeton parvum*, from these beds; and the following year he described (Proc. U. S. Natl. Museum, XL, p. 429-433, 1911) and figured further remains of the same species and described a new microsaurian as *Amphibamus thoracatus*.

There have been, thus, up to the present time, four species of Amphibia described from the Mazon Creek shales. These four species are represented by seven specimens. The type of *Amphibamus grandiceps* Cope was destroyed by fire, but there is an excellently preserved specimen of this species in the collection of Mr. L. E. Daniels, of La Porte, Ind. Other examples of the fossil Amphibia of Mazon Creek which have come to the writer's notice are specimens, representatives of the Branchiosauria and the Microsauria, in the collection of Mr. R. D. Lacoe, now the property of the U. S. National Museum. This small but highly interesting collection has recently been loaned the writer for study by Mr. Charles W. Gilmore. This was the sum total of Mazon Creek Amphibia known to the writer until some months ago. In November of 1909 a collection of nodules containing Amphibia was loaned the writer for study through the courtesy of Doctors Eaton and Schuchert, of Yale University. This lot consists of ten individuals, representing seven genera and eight species, all of which, save one, are regarded as unknown and have been described as new. This is an immense addition to our knowledge of the amphibian fauna of the Mazon Creek shales and adds much to our knowledge of the diversity of structure displayed by the Amphibia of the Carboniferous.

The forms described below are entirely unlike any of the forms occurring elsewhere in the Carboniferous or later ages. It has been the writer's good fortune during the past five years to examine all of the specimens of Carboniferous air-breathing vertebrates in North America, with the exception of Sir William Dawson's collection at McGill University. The European forms, as they have been described from time to time, are, unfortunately, known to him only through the literature, with the exception of a small collection of Branchiosauria recently received in exchange from Dr. Hermann Credner. It is hoped that in the near future this may be supplemented by actual observation, and until that time it can not be positively asserted that the forms described below are unlike those already known, but, so far as his knowledge goes, the writer is confident that they are new to science. The characters used for generic distinctions are such that even the most superficial observations must reveal. They are structural ones, and are those which are used by many eminent vertebrate paleontologists at the present time. Unfortunately, we know so little about the development of the class Amphibia that we can not always be sure that our characters are phylogenetic, as they must be to mean anything. So that until some idea of phylogeny is obtained, structural characters must be used which seem to the describer to be of generic significance.

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[1] Eastman, C. R., Journ. Geol., vol. 10, p. 535, 1902.

[2] Moodie, Roy L., Journ. Geol., vol. 17, p. 39, 1909.

The Amphibia so far discovered in the Mazon Creek shales, including those described in this paper, are:

1. *Amphibamus grandiceps* Cope, 1865.
2. *Amphibamus thoracatus* Moodie, 1911.
3. *Micrerpeton caudatum* Moodie, 1909.
4. *Eumicrerpeton parvum* Moodie, 1910.
5. *Mazonerpeton longicaudatum* Moodie.
6. *Mazonerpeton costatum* Moodie.
7. *Cephalerpeton ventriarmatum* Moodie.
8. *Erpetobrachium mazonensis* Moodie.
9. *Spondylerpeton spinatum* Moodie.
10. *Erierpeton branchialis* Moodie.

These ten species are distributed among eight genera, five families, and four orders, thus showing the amphibian fauna of Mazon Creek to be a diverse one. The arrangement of the species into groups is given below.

Class: AMPHIBIA, Linne, 1758.

Subclass: Euamphibia, Moodie, 1909.
  Order: Branchiosauria, Lydekker, 1889.
    Family: Branchiosauridæ, Fritsch, 1879.
      Micrerpeton caudatum Moodie.
      Eumicrerpeton parvum Moodie.
      Mazonerpeton longicaudatum Moodie.
      Mazonerpeton costatum Moodie.
  Order: (?) Caudata, Duméril, 1806.
    Family: Cocytinidæ, Cope, 1875.
      Erierpeton branchialis Moodie.
Subclass: Lepospondylia, Zittel, 1887.
  Order: Microsauria, Dawson, 1863.
    Family: Amphibamidæ, Cope, 1875.
      Amphibamus grandiceps Cope.
      Amphibamus thoracatus Moodie.
      Cephalerpeton ventriarmatum Moodie.
    Family: Molgophidæ, Cope, 1875.
      Erpetobrachium mazonensis Moodie.
Subclass: Stegocephala, Cope, 1868.
  Order: Temnospondylia, Zittel, 1887.
   Suborder: Embolomeri, Cope, 1885.
    Family: Cricotidæ, Cope, 1884.
      Spondylerpeton spinatum Moodie.

The discovery of the embolomerous amphibians in the Carboniferous fauna is not new, since the first embolomerous form known in North America was described from the deposits on Salt Creek, Illinois, as *Cricotus heteroclitus*, by Cope. Later the same or a closely related form was discovered in Texas by Cope and Case and in Kansas by Williston. The form described here is, however, much more primitive than any of the species of *Cricotus*. The rachitomous forms of Amphibia are known from the Carboniferous of North America and Europe through the researches of Fritsch and Case.

The content of the amphibian fauna of the Mazon Creek shales is peculiar on account of the presence of the four species of Branchiosauria. Unless Dawson's *Sparodus* is an example of this group, the forms in the Mazon Creek fauna represent the only known occurrence of this order in North America. Dawson was himself doubtful about the identity of the remains which he referred to *Sparodus*. Judging from his figures, there is a possibility that he may be right, since the form of the interclavicle is decidedly branchiosaurian, as we know the form of that element among the European species. Beside the presence of the Branchiosauria, the Mazon Creek amphibians differ from the Canadian species in the almost total absence of any scaly covering such as occurs in *Dendrerpeton* and *Hylerpeton,* although *Micrerpeton* has very small scales over the body and tail. Furthermore, the Canadian species are more terrestrial than those from Mazon Creek, which, judging from their form, were either entirely aquatic or only partially terrestrial. The size of the members of the two faunas differs in no great degree. Large and small members are found in both localities. The Joggins Amphibia are hardly well enough known to judge their relationships other than those of an ordinal or family rank. These relations will be given in another paper.

The Linton fauna, which is more fully described in another paper, is quite unlike the Mazon Creek fauna. This is evident by the absence of branchiosaurian forms from the Linton deposits and by the presence of the legless Microsauria and the Proteid form *Cocytinus,* which is paralleled by *Erierpeton* in the Mazon Creek fauna. The limbed Microsauria also differ in a marked degree, in that the Mazon Creek forms have a strong tendency toward the Reptilia, as illustrated in the Amphibamidæ, *Amphibamus* and *Cephalerpeton.* The Linton fauna shows a wide divergence of types, illustrating different phases of amphibian development, and in the tendency of certain groups, such as the families Tuditanidæ and Macrerpetidæ, to approach the Stegocephala proper and through them certain of the stegocrotaphous reptiles. The Linton fauna is distinctive too in the abundant presence of such highly developed swimmers as *Œstocephalus, Ptyonius, Phlegethontia* and *Ctenerpeton,* and as such is interesting in displaying parallel development of the same structures in forms which are really widely separated in structure.

The fauna of the Cannelton slates is not very different from that of Linton, and the remarks made concerning the relationship of that fauna to the Mazon Creek fauna will also apply with reference to the Cannelton fauna. Outside of these four deposits, there are several minor deposits which have furnished amphibian remains, none of which agree in any essential respect with the Mazon Creek fauna. Attention has already been called to the fact of the occurrence of the embolomerous forms in the Mazon Creek, the Pitcairn, Pa., the Kansas, and the Salt Creek, Illinois, faunas.

A view of the entire amphibian fauna of the North American Carboniferous, as we know it at this time, shows us that the separate faunas were local, and as such indicate the ancient history of the group at that time. Such a high degree of developmtnt and such a wide dispersal of types would indicate a long antecedent history. Possibly the Amphibia of the Mississippian rocks will yield forms which will connect these local faunas; possibly we may have to look to the Devonian for these connections. The early Mississippian and Devonian forms are already indicated by footprints, but as yet we know nothing of the structure of the creatures which made the footprints.

DESCRIPTIONS OF GENERA AND SPECIES.

Genus ERIERPETON—New Genus.

All of the examples of Amphibia loaned the writer by the Yale University Museum are capable of identification. One of the most unusual forms is represented by a distinct impression on a weathered ironstone nodule from which all the bony matter had become eroded. It is so unusual in form and in the characters which it presents that it is deemed worthy of description. Since it is totally unlike anything described, it must be placed in a new genus, for which the term *Erierpeton* is proposed. The name refers to its early appearance. The specimen in question is No. 801 (222) 5 of Yale Museum. The nodule which contains the impression is some three inches in long diameter.

The generic characters are found, first of all, in the presence of hyobranchial arches, which indicate its relationship to the formerly described *Cocytinus gyrinoides* Cope, from the Carboniferous of Ohio. The only other known extinct genera of Caudata which possess, or at least have preserved, the branchial arches are the Jurassic *Hylæobatrachus* from Belgium and *Lysorophus* from the Permian of Texas. The present form is widely distinct from both of these genera in the shape of the mandible and the form and arrangement of the hyobranchial arches. The new genus finds its closest ally in *Cocytinus* in the family Cocytinidæ which possibly belongs in the order Caudata and in the suborder Proteida of Cope.

Ericrpeton branchialis, New Species.

(Plate 1, fig. 3; plate 2, fig. 1.)

The amphibian remains which are designated by the above name consist of a distinct mandible and some rather indefinite body impressions. Three elongate impressions occur between the rami of the mandibles, which, I suppose, must represent hyoid bones belonging to the branchial arches. The lateral elements are paired and the median impression is straight and lies between the paired portions of the hyoids. The paired portions probably represent the hypohyals, or hypohyals plus the ceratohyals, and the unpaired portion the first basibranchial, according to the nomenclature of Wiedersheim (Comparative Anatomy of Vertebrates, 1897, p. 86). If the impressions have been correctly interpreted, the present specimen is of very great interest since it is the first evidence we have of the hyobranchial arches in the Amphibia of Mazon Creek, and the second in the Carboniferous of North America. Dawson doubtfully identified some elements of the Joggins Amphibia as hyoids, but was uncertain as to their position. Cope described fully the well-developed hyobranchial apparatus of *Cocytinus gyrinoides* from the Coal Measures of Ohio. Among other Paleozoic Amphibia, Williston has described branchial arches in the peculiar form *Lysorophus tricarinatus* Cope, from the Permian of Texas.

The form of the impression of the mandible in the present specimen is unlike anything known to the writer among other Carboniferous or later Amphibia. The rami are long, slender, deep, slightly curved and pointed anteriorly. The anterior symphysis was not a complete sutural union, but was occupied partly by cartilage or connective tissue.

There are no definite traces of appendicular structures. The traces of the body, plate 2, fig. 1, indicate an elongated, rather slender animal, but further than that nothing can be said in regard to its structure.

The occurrence of a typically caudate form in the Carboniferous is unusual, and complicates still further our understanding of the origin and relationships of the early Amphibia.

MEASUREMENTS OF THE TYPE OF ERIERPETON BRANCHIALIS MOODIE.

	mm.
Length of entire impression	50
Length of mandible along median line	10
Width of mandibular ramus	9
Length of basibranchial	2.5
Width of basibranchial	.75
Length of hypohyal	2.4
Width of hypohyal	1.5

Genus MICRERPETON, Moodie, 1909.

This genus was established by the writer on a single excellently preserved specimen (No. 12,313, University of Chicago). The genus is readily distinguished from other members of the Branchiosauridæ by the large size of the orbits, the short, heavy limb bones, the slender ilium and the expanded, elongate and flattened tail.

Micrerpeton caudatum Moodie, 1909.

There is but a single species known. It is fully described in previous papers. (Journal of Geology, vol. xvii, No. 1, pp. 39-51, with seven figures, 1909; Journal of Morphology, vol. xix, No. 2, pp. 516-520, with three figures, 1908.)

Genus EUMICRERPETON, Moodie, 1910.

Amer. Natl., vol. XLIV, June, 1910, p. 367.

This genus is based on well-preserved remains of three individuals from the Mazon Creek shales. One of the specimens represents, apparently, an adult, and the others are immature. The manner of the impressions resembles in a marked degree those described by Thevenin from the Commentry beds of France (plate 14). The nodules which inclose the remains measure, respectively, two and one-quarter and two and one-half inches in long diameter.

The generic characters are found in the very broad posterior table of the skull, with its short longitudinal length, the reduction of the tympanic notch and the short length of the body. The body length of *Eumicrerpeton* is as 2 to 4, while that of *Micrerpeton* is as 2 to 5, and that of *Branchiosaurus fayoli* is as 2 to 4⅓. The entire impression of the branchiosaurians from Saxony are not preserved, so that comparative measurements are not possible. Other generic characters are found in the sharp supratemporal angle of the skull, and it is to be distinguished from *Micrerpeton* especially by the short, stumpy limb bones. Its distinctions from the genera of European Branchiosauria are the same as those which distinguish it from *Branchiosaurus,* to which it is closely allied. The narrow, elongate eye, placed close to the edge of the skull, is a character not observed hitherto in the Branchiosauria. It recalls the condition described by Credner for the young forms of *Branchiosaurus amblystomus* Cred.[3]

Eumicrerpeton parvum Moodie, 1910.

(Plate 3, figs. 3 and 4; plate 4; plate 5, fig. 1; plate 6, figs. 1 and 2.)

The specific characters are found in the anatomical details. The impression of the outline of the entire body is preserved in both animals, and in both are found impressions and molds of the alimentary tract which in the younger animal are remarkably complete and instructive.

The impression of the larger animal, which is probably an adult, presents the following elements: the entire skull; both humeri; impressions of posterior and anterior ventral armature; portions of the alimentary canal; one femur; portions of a fibula and tibia; and the entire impression of the tail, on which, as in *Micrerpeton caudatum* Moodie, there occur two definite dark lines, one beginning at the tip of the tail and running obliquely along the tail to where the impression is broken at the anal region; the other beginning at a distance of four and one-half millimeters from the tip and running almost parallel with the median line. These two lines undoubtedly represent the lateral line system.

The skull is especially noted for its shortness and the great posterior width, as well as for the almost entire absence of the tympanic notch. The pineal foramen is located on a line with the posterior border of the orbits. The eyes themselves are narrow and acuminate at each end, with a pronounced convexity inward and a flattening outward. They are located on the very border of the skull, but relatively further posterior than in *Micrerpeton.* No sclerotic plates are evident. The median suture can be indistinctly observed running the entire length of the skull. The sutures bounding the outside of the frontals and the squamosals are partially evident but not satisfactorily preserved. The mandible is represented by a mold which in a wax impression shows short, stumpy teeth.

Posterior to the skull, at a distance of one millimeter, there are two sharp impressions, which may represent the anterior edges of the interclavicle, or they may be branchial elements. They are distinctly curved, however, and probably represent portions of the interclavicle. A wax impression does not show a discrete structure, but the boundaries of some larger element. No other remains of the pectoral girdle can be discerned. The humeri are short and relatively thick. Wax impressions show them to have had truncate or slightly concave ends, thus indicating the absence or slight development of endochondrium. No other elements of the arm are preserved.

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[3] Zeitschrift d. Deutsch. Geol. Gesellschaft, 1886, Th. VI, Taf. XVI, fig. 1.

The ventral armature is preserved in two small patches, and these show the chevron-shaped rods to have been very fine—much more delicate than in *Micrerpeton.*

The body impression is very interesting, both as showing the form of the body and because in it are preserved the impressions of the larger portion of the alimentary canal. The form of the body can best be discerned by reference to the figures. (Plate 3, figs. 3, 4; plate 6, figs. 1, 2.)

The portions of the alimentary canal preserved consist of the greater portion of the stomach, three coils or loops of the small intestine, the rectum, and a pit which undoubtedly represents the anal opening. The anus is found at a distance of 16 mm. from the tip of the tail, and is somewhat removed from the body portion, as in modern salamanders. On each side of the posterior end of the rectum there occurs a pair of enlargements, which probably represent the oviducts at their posterior extremity.

The tail impression is more acuminate than in *Micrerpeton,* but shows the same structures as that form, *i. e.,* the lateral lines, which have already been mentioned. *Micrerpeton* was a more rapid swimmer than the present form, on account of the greater development of the tail. The impression of an elongate femur and the heads of the tibia and fibula of the left side are preserved.

The second specimen of the species (No. 802, Yale Museum) shows much the same characters as the specimen already described, except that there are preserved impressions of small, blunt teeth on the mandible. The two humeri and the femur of the left side are preserved, and the interclavicle is represented by an identical impression as in the first described specimen. The tail impression, although similar in form, does not exhibit so much of the structure of the lateral lines. The present specimen is considered as more immature than the former, on account of its smaller size. There are no positive evidences in either specimen of branchial arches.

The matter of especial interest in connection with the second specimen is the remarkably perfect preservation of the alimentary canal. It is entire except for the very anterior end of the œsophagus. The posterior portion of the œsophagus, which measures three and one-half millimeters, is clearly preserved. Its anterior end is thrown around posteriorly, and indicates that this end was loosened after death and became displaced before preservation. The length preserved possibly represents the entire œsophagus. The œsophagus is constricted before it enters the stomach, which shows the usual curvature found in modern salamanders. The stomach measures six millimeters in length by two in greatest breadth. It consists of a single enlargement, as in the modern *Ambystoma punctatum.* The stomach enlarges somewhat toward the pyloric end, and then very gradually constricts to the pyloris. Three diameters of the small intestine can be discerned. The most anterior one, corresponding with the duodenum, is segmented, as though the intestine were filled with food substance. The remainder of the intestine, corresponding to the ileum, is looped in the form of two figures "8," which are superimposed, with the upper portions of the "8" at right angles to each other. The rectum is clearly discernible, though its lower end is somewhat obscured by having the lower portion of the upper loop of the intestine lying over it. The anus lies at a distance of one and one-half millimeters posterior to the line from the upper end of the femur, and is quite well back on the tail, as in modern salamanders. In this specimen also occur two oval bodies, which may be identified as the lower ends of the oviducts; thus indicating, in all probability, that the animal was a female.

A dissection of several species of modern urodeles has resulted in the discovery that the adult condition of the alimentary canal of all species dissected—*Ambystoma punctatum, Necturus maculosus, Diemyctylus torosus,* etc.—is much more complex than that exhibited by the specimen under discussion. A very near approach to the condition found in *Eumicrerpeton parvum* is found in an immature branchiate individual of *Diemyctylus torosus,* 56 mm. in length, from a fresh-water pond on Mount Constitution, on Orcas Island, Puget Sound, Washington.

The similarity of intestinal structure is of considerable importance to our understanding of the relationship existing between the Carboniferous Branchiosauria and the modern Caudata. This fact only confirms other arguments, offered in another place, concerning their immediate relationship. (American Naturalist, vol. xliv, June, 1910.)

The branchiosaurian affinities of the present species are almost too evident to need discussion. The entire structure is essentially similar to that of other genera of the order.

MEASUREMENTS OF EUMICRERPETON PARVUM MOODIE.

Specimen No. 803, (222) Yale University Museum.

	mm.
Length of animal	37.5
Length of skull	4.5
Posterior width of skull at table	6
Long diameter of eye	1.75
Transverse diameter of the eye	.65
Length of left humerus	1.5
Length of femur	1.75
Width across base of tail impression	3.5
Length of tail from base to tip	17
Number of ventral armature rods in 1 mm	10

Measurements of second specimen, No. 802, (471) Yale University Museum.

	mm.
Length of animal	30
Length of skull	4
Posterior width of skull	5
Length of œsophagus	3.5
Length of stomach	6
Width of stomach	1.33
Estimated length of intestine	18
Width across base of tail impression	2.5
Length of tail from base to tip	7

Eumicrerpeton parvum Moodie (an additional specimen).

(Plate 5, fig. 1.)

After the above had been written the writer received from Mr. C. W. Gilmore, of the U. S. National Museum, an additional specimen of this species. It is No. 4400 of the U. S. National Museum. The additional specimen serves to substantiate the above-described genus and species, and shows more clearly characters which are distinct from *Micrerpeton*, the genus to which the present form is most nearly related.

The present specimen is almost as perfectly preserved as was the specimen of *Micrerpeton caudatum* Moodie. When the nodule containing the fossil was received the tail was embedded in matrix, but by careful use of chisel and hammer it was possible to lay bare the whole tail, the tip of which ends on the very edge of the nodule. This was at once perceived to be precisely similar to that of the above-described specimens. The skull structure, the intermediate position of the pineal foramen, the epiotic notch and the shape of the skull are so exactly similar to those of *Eumicrerpeton parvum* that the specimen was unhesitatingly referred to that species.

Most interestingly, too, the present specimen has the alimentary canal preserved almost as perfectly as in the other two specimens; so that the three specimens of this species now known show the alimentary canal. The present specimen is, however, much more developed than the other two, if we may judge from the relative sizes. There is not the slightest trace of branchiæ in any of the specimens. The matrix does not preserve the skeletal elements as well as does the hard dolomite from Saxony, in which Doctor Credner found such excellently preserved branchiæ.

The present specimen is nearly half again as long as the smallest of the above-described specimens, and the skull is proportionately longer and wider. There is preserved also an impression of the anterior edge of both clavicles, as has been described for the Yale specimens; no other portion of the pectoral girdle is preserved. The right humerus is imperfectly preserved, as is also the right femur and tibia; other than these the fossil is merely an impression.

The skull is so nearly like what has been described for the Yale specimens that additional description is unnecessary. The pineal foramen is quite large, and lies on a line which cuts the orbits into equal longitudinal parts. The interorbital space is about equal to the long diameter of the orbit, as in the Yale specimens. Traces of sclerotic plates are observed in the left orbit, but they are quite imperfect.

The alimentary canal is unlike that of the Yale specimens, in that the INTESTINE is longer and much more convoluted. It lies in five longitudinal folds and ends in an enlarged cloaca, near which there are impressions of two glands, or they may be the posterior ends of the oviducts, as was suggested for the Yale specimens. Like the Yale specimens, the œsophagus is displaced and partially obscured. The creatures undoubtedly fed on small plants and animals, much as do our recent salamanders. The alimentary tract is preserved fully extended.

MEASUREMENTS OF ADDITIONAL SPECIMEN OF EUMICRERPETON PARVUM MOODIE.

(Cat. No. 4400, U. S. N. M.)	*mm.*
Length of entire animal	45
Length of skull	6
Width of skull	9
Transverse diameter of orbit	1.50
Long diameter of orbit	2.25
Interorbital space	2.50
Diameter of pineal foramen	.50
Length of body from back of skull to pelvis	22
Greatest width of body	9
Length of tail	16
Width of tail at base	5
Length of humerus	3
Length of femur	2.50
Length of tibia (fibula ?)	1.75
Length of stomach	7
Width of stomach	3
Length of intestine (estimated)	56
Width of intestine	1

Genus MAZONERPETON, new genus.

It was very gratifying to discover among the remains loaned the writer by the Yale Museum other specimens exhibiting characters of the Branchiosauria, for our knowledge of this order of Amphibia is as yet very incomplete in North America. The specimen represents by far the largest of the group discovered on this continent. It is more than twice as long as the specimen of *Eumicrerpeton parvum* and fully one-third longer than *Micrerpeton caudatum*. It is, however, distinctly a branchiosaurian. The ordinal characters are discovered in the heavy, straight ribs, attached to the transverse process of the centrum; in the low degree of development of the vertebræ; in the structure of the skull and the ventral armature, and in the degree of ossification of the limb bones.

It may be generically separated from other known Branchiosauria found in North America by the great length of the dorsal region, and by the elongate tail with its well-developed caudal ribs. It is not so readily separable from the Branchiosauria of Europe. It is most closely related to *Branchiosaurus amblystomus* Credner of the Permian and Carboniferous of Saxony. From this genus, however, *Mazonerpeton* may be distinguished by the reduction of the posterior tympanic notch, the broad nature of the scapula, the elongate interclavicle and the slender ilium in the present form. The number of dorsal vertebræ is identical in the two genera.

The genus is so closely allied to *Branchiosaurus* of Europe that the two species here described must be located in the family Branchiosauridæ.

Mazonerpeton longicaudatum new species.

(Plate 3, figs. 1 and 2; plate 7, fig. 3; plate 10.)

The remains on which the above species is based consist of the following elements: an incomplete skull, nearly the entire vertebral column, consisting of cervical, dorsal, sacral and caudal vertebræ, 36 in number; several ribs preserved on each side of the vertebral column, a portion of the ventral armature, the scapulæ, a clavicle, the interclavicle, both humeri, the radius and ulna of one side and the ulna of the other, portions of both hands, the ilium of the right side, both femora, and a partial impression of the left tibia.

The skull is unfortunately very poorly preserved. Enough remains, however, to determine the essential characters. The skull bones, unlike any other American branchiosaurian, have an ornamentation consisting of sharp pits and elevations, which in places have a quincuncial arrangement and in others take the form of definite lines of pits or tubercles similar to the condition found in many of the Microsauria. The orbits are large and are situated back of the median transverse line of the skull. They are almost circular in form and contain six elongated sclerotic plates very closely arranged around the borders of the right orbit. The plates are twice as long as wide. The interorbital width is one and one-fourth as great as the transverse diameter of the orbit.

Not many of the sutures of the skull are discernible. Portions of the frontals, the nasals, the prefrontals, the parietals and the supratemporals can be identified. Their arrangement is shown in figure 3, plate 7. There is a decided posterior table to the skull, with a truncate posterior border. The tympanic notch is shallow, with its outer border not so well protected as in *Branchiosaurus.*

The cervical vertebræ are incomplete, but their number was four or five, as in *Micrerpeton.* The structure of the dorsal vertebræ is also uncertain, although the shape can be discerned. The vertebræ are short and thick, very unlike the long, cylindrical vertebræ of *Cephalerpeton.* The heavy transverse process is quite evident on the best preserved vertebræ. This process recalls that described by Credner for the Saxony Branchiosauria. Several of the vertebræ show the attachment of the ribs to this process. The ribs of the caudal region recall very strongly those of *Branchiosaurus.* They are quite heavy in the anterior caudal region and then diminish rather rapidly to the point where the tail is broken and lost.

The ventral armature is represented by a patch of chevron rods twenty-one millimeters in length. The rods take a very peculiar form. They are short, crescentic bundles of fine rods, hair-like in appearance. In one of the bundles I count five smaller rods. The bundles are arranged in rows similar to the pattern so characteristic of the Carboniferous Amphibia, as described elsewhere. The patch of ventral armature preserved belongs to the abdominal region, so nothing can be told of the gular and thoracic rods. A single row of the crescentic bundles measures 11 mm.

Both scapulæ are preserved in their entire form. They are quite different from those of any other genera. They resemble a broad crescent with a posterior concavity and an anterior protuberance. The articular surface of both scapulæ is obscured. Vascular foramina occur near the base of both scapulæ. There are three of them in the right element, arranged in the form of an isosceles triangle. The morphology of these three foramina is uncertain. They have never before been observed among the Carboniferous Amphibia, and, so far as I am aware, they are entirely unknown among the higher vertebrates.

The temnospondylous Amphibia of the Carboniferous and Permian possess, in the coössified scapula-coracoid, three foramina, very similar to the present ones, but they are confined to the coracoidal region, and in the Branchiosauria the coracoid, as identified by Credner, is a free element, although I have never been sure with regard to its identity among American forms. Williston, in *Trematops,* has called these foramina the glenoid, supraglenoid and supracoracoid foramina

(Journ. Geol., xvii, No. 7). These are not, however, to be correlated with the three foramina above mentioned, since in the Temnospondylia the foramina belong with the coracoid and not with the scapula. The condition of the Temnospondylia occurs in the bony fishes, *Xiphactinus audax* Leidy; and an analogous condition obtains in the reptiles, as in the mosasaurs and dinosaurs, where the separate coracoid is pierced by foramina. Doctor Williston informs me that the foramina are also found among the Cotylosauria, where the condition is not far different from what it is in *Eryops*.

Near the outer end of the right scapula there is a large fragment preserved which, I think, must be the misplaced clavicle. It is obscurely triangular, or, more exactly, spatulate. The interclavicle is represented by fragments only. It seems to have had a narrow form.

The humeri recall those of *Micrerpeton*. They are somewhat elongate and apparently cylindrical in their normal condition, though somewhat flattened in the fossil. The shaft is considerably constricted at the middle, and the ends are expanded, in which expansion the lower end exceeds. The ends are abruptly truncate, indicating a small amount of endochondral ossification or its entire absence.

The mesopodial elements, unlike what has been described for *Cephalerpeton*, are quite dissimilar in form, recalling the condition in *Mesosaurus brasiliensis* McGregor. The larger element I take to be the ulna. It has the lower end greatly expanded and the shaft is curved outward. It resembles very much a reptilian ulna.

The radius is much smaller than the ulna, lacks the lower expansion, and is shorter by one millimeter. Its ends are abruptly truncate.

The carpus is represented merely by a blank space. There are no evidences of impressions of cartilage in the sandstone. The hand of the left side contains four digits. There are two phalanges preserved in the first digit, including the sharp-pointed terminal phalanx. The second digit has only the metacarpal. The third has the metacarpal and the first phalanx, which does not differ in form, but only in size, from the metacarpal. The fourth digit contains only the metacarpal. No definite evidence of more than four digits has ever been given for the hand of the Branchiosauria. Of the right hand there are portions of three digits preserved, including three metacarpals and one phalanx. In structure they are not different from those of the right hand.

The ilium of the left side is preserved, apparently entire. It is elongate and cylindrical. Its upper end lies adjoining the twenty-eighth vertebra.

The head of the femur lies close to the lower end of the ilium, so that that element must have been suspended in the flesh much as in the modern salamanders. It could not have been of much use in support. The form of the femur is not unlike that described for the humerus, save that its lower end is smaller than the upper, while in the humerus both extremities are alike. A portion of the right femur is preserved extending in an opposite direction to the left. No portions of the leg or foot are preserved.

MEASUREMENTS OF THE TYPE OF MAZONERPETON LONGICAUDATUM MOODIE.

	mm.
Length of entire specimen....................	64
Length of portion of skull preserved...........	6.5
Posterior width of skull preserved............	7
Width across orbits..........................	11
Long diameter of the orbit....................	3
Transverse diameter of orbit.................	1.75
Interorbital width	1.75
Length of dorsal vertebræ....................	48
Length of caudal series......................	11
Length of anterior dorsal vertebra; 1 centrum..	2
Length of anterior dorsal rib.................	4
Length of anterior caudal rib................	1.75
Length of scapula...........................	5
Greatest width of scapula....................	4.25
Probable length of interclavicle...............	6
Width of interclavicle........................	3
Length of clavicle...........................	4.5
Width of clavicle............................	1.5
Length of right humerus.....................	6
Distal width of humerus.....................	2
Length of ulna..............................	3.25
Distal width of ulna.........................	1
Length of radius............................	3
Width of carpal space.......................	2
Length of metacarpal........................	1.74
Length of first phalanx......................	1.75
Length of distal phalanx of right hand.........	.35
Number of bundles of chevron rods in 1 mm....	4
Length of ilium.............................	2.25
Length of femur.............................	4
Proximal width of femur.....................	1.50

The type is specimen No. 795 (1234), with obverse, of the Yale University Museum. Collected at Mazon Creek, Grundy county, Illinois.

Mazonerpeton costatum new species.

(Plate 2, fig. 3; plate 8, fig. 4; plate 9, fig. 2; plate 10.)

The remains on which the present species is based are inclosed in a much-fractured nodule. The parts of the animal which have been identified are as follows: A part of the skull and left mandible, two clavicles, a humerus, impressions of several vertebræ, a portion of the dorsal region of the body, with several ribs, two portions of the caudal region, with several ribs, and some unidentified fragments.

The animal, from the shape and form of the ribs, is undoubtedly a representative of the Branchiosauria, since short, heavy, straight ribs have not yet been found to be associated with other than branchiosaurian structures. Its association in the same genus with *Mazonerpeton longicaudatum* is held to be correct, on account of the resemblance in structure of the pectoral elements, the form of the humerus, and the length of the tail. The present species is about one-half larger than *Mazonerpeton longicaudatum,* and the animal which represents the species perhaps attained a length of four and one-half inches, while the length attained by the type of *M. longicaudatum* was not more than three inches. The tail of the present species is very long and slender, more elongate than in any other described branchiosaurian.

The part of the skull preserved is very unsatisfactory, and, aside from the fact that it seems to represent the under side of the left half of the skull, little can be said. Portions of three sutures can be observed, but what sutures they are is undetermined. The left mandible lies crushed on the edge of the skull and partially obscures what little there is of that structure. The slightly curved impression, from which the bone has been either broken or weathered, measures thirteen millimeters in length by three in posterior diameter by one in anterior diameter. These measurements show the element to have been slender and pointed anteriorly.

Very little accurate information can be derived from the study of the vertebral column of the specimen. The dorsal vertebral formula can not be made out, since only a portion of the length of that region is preserved, and only a few rather indefinite impressions can be discerned. These impressions show the vertebræ to be short and higher than in most Branchiosauria.

The caudal series is represented by two sections. One of these sections is apparently from near the base of the tail, judging from the size of the caudal ribs preserved. The other section is from near the tip of the tail, and it shows the constituents to have been long and slender. Ribs are apparently absent on this section. The position of the two caudal sections shows that when the animal died it was coiled up much like a snake, so that in the fractured nodule three sections of the body are preserved. The tail was probably half as long again as the body.

The ribs throughout the body are short, heavy and straight, with, in the dorsal series, a lateral and a distal expansion, which is taken as a distinctive specific character. Judging from imperfect impressions in the dorsal series, the ribs were attached to a transverse process of the centrum, thus agreeing with other branchiosaurians in this respect. The ribs show a progressive decrease in length from the cervical region to the point of their disappearance on the tail.

The pectoral girdle is represented by two elements, one of which is certainly the right clavicle, and the other is possibly the left clavicle, though its form is somewhat distorted by pressure. Both elements are in the form of an elongate spatula, with the dorsal surface greatly concave and the inner end acuminate.

The right humerus is imperfectly preserved, though the impression allows one to gain an exact knowledge of its form. It lies under the right clavicle. Its ends are truncate with a contracted shaft and expanded extremities. The bone was apparently hollow.

In another nodule (No. 804, Yale Museum) there is a single bone preserved, which resembles to a great extent a rib of the present species, although somewhat larger, and it has been provisionally identified as such. The element is very slightly curved, but it shows the expanded head of this species. (Plate 2, fig. 3.)

Measurements of the Type of Mazonerpeton costatum Moodie.

	mm.
Length of portion of skull preserved	14
Length of right clavicle	16
Width of right clavicle	4
Length of dorsal region represented	30
Length of cervical rib	8
Length of dorsal rib	6.5
Length of caudal rib	3
Length of caudal portion of the body preserved	55
Length of mandible	15
Greatest width	6
Length of right humerus	10
Greatest width of right humerus	2

The type is No. 800 (777) of the Yale University Museum. Collected at Mazon Creek, Grundy county, Illinois.

Measurements of Specimen No. 804 (332).

	mm.
Length of rib	11
Width of head of rib	2
Diameter of shaft	1

Amphibamus grandiceps Cope, 1865.

(Plate 1, figs. 1 and 2; plate 5, fig. 3; plate 7, fig. 1; plates 11, 12, 13.)

The collection of Carboniferous Amphibia loaned the writer for study by the Yale Museum contains an unusually perfect example of *Amphibamus grandiceps* Cope. The skull is nearly complete, although the sutures are indistinct. The following parts have been identified in the specimen: the greater part of the vertebral column, ventral armature, ribs, portions of the pectoral girdle, the pelvic girdle, and all four limbs, with the hands and feet in an unusually perfect condition, all very clearly and distinctly shown on a nodule from the Mazon Cheek shales of Illinois. The specimen was collected near Morris, Illinois, in 1870.

The writer (1909) published a restoration of this species, in which he gave to the vertebral column twenty-six vertebræ, the exact number being at that time uncertain. Professor Cope (1865) in his original description gave the number as thirteen between the interscapular region and the sacrum. Hay (1900) thought the number was less than twenty. The present specimen shows that there were twenty-two in the presacral region, not including the sacral vertebra; thus showing that in two cases too few vertebræ and in the third case too many vertebræ were assigned to the vertebral column.

The author's published restoration gave too great a length of tail. The present specimen shows only ten cadual vertebræ, the most anterior of which are provided with short ribs.

One of the most interesting features of the present specimen of *Amphibamus grandiceps* Cope is the preservation of a small patch of skin, evidently from the back. It lies off to one side, near the head, as though the skin had been loosened and floated away from the body or was moved in some manner. The remnant measures 5 mm. in length by 3 mm. in width. The fragment shows the skin to have been made up of tuberculated scales, four of which occur in the length of one millimeter. The scales are somewhat hexagonal, almost rounded, and were relatively quite thick. They lie in a close mosaic.

The cranial structure presents no new features. There are evidences of twenty small, oblong, sclerotic plates preserved in the right orbit. These form about two-thirds of the circumference of the iris, so that twenty-nine or thirty was probably the correct number of these plates. Their position near the center of the orbital space shows clearly that they were sclerotic plates, and not palpebral scales, as Professor Cope thought they might be from his study of the type. The obverse of the specimen shows that the skull bones were pitted, especially in the nasal region, as Hay has described for the specimen in the possession of Mr. Daniels. The sutures are very indistinct and uncertain and can not be described. They are well known, however, in other specimens.

The present specimen adds to our knowledge of the ventral scutellæ, as is shown in figure 1, plate 7. The plates of the throat, chest and belly have different directions. The arrangement of the plates on the throat and chest is almost exactly the reverse of what Credner has described for *Branchiosaurus amblystomus* Cred. On the throat, in the present form, the chevron points anteriorly, and it is the anterior prolongation of the belly scutes with the posterolateral projection of the gular plates which forms the chest and arm scutellation. The belly chevrons point anteriorly, as in the species from Saxony. The rods formed by the scutes are straight, and not curved as in *Branchiosaurus*. The entire ventral armature preserved is misplaced to the left of the animal, and only the anterior portion is preserved, containing a length of 18 mm. There are three scutes to the millimeter.

Nothing new can be ascertained of the structure of the pectoral girdle. Portions of the scapulæ, clavicles and interclavicle are represented, and perhaps the coracoid is indicated by a fragment.

The specimen in hand completes in a very satisfactory manner our knowledge of the structure of the pelvis. The relations, form and structure of the ilium, ischium and pubis are now known quite certainly. The form of the ilium as shown in previous restorations was slightly inaccurate. It was made a little too long and too curved. It is, instead, rather short and nearly straight, with the ends expanded. Heretofore nothing was known about the ischium, but the present example shows very clearly the form of that element on both sides of the vertebral column. Its form is almost identical with that of *Paleohatteria longicaudata* Credner, from the Rothliegenden of Saxony. They are apparently approximate in the median line, though this character is somewhat obscured by the impressions of the caudal vertebræ. Its relation with the ilium, other than that it was posterior to it, is uncertain. All three of the pelvic elements were undoubtedly hung loosely in the flesh of the animal, as in modern salamanders, since none of the elements present any marked articular surfaces.

The structure of the sacral vertebra and the sacral ribs still remains to be determined. There seems to have been but a single pair of sacral ribs, but the specimen is too obscured to shed much light on that point.

Nothing new is added to our knowledge of the arm. The number of phalanges can not be ascertained. Two of the digits are preserved entire, and there is nothing in their structure to contradict the number given in the restoration which is herewith republished. Each hand has a single digit of four segments preserved. They undoubtedly represent the third digit in each case.

The elements of the leg and foot are as they are given in the restoration. The right foot is preserved almost entire, and the digits have the formula 2-2-3-4-3. The distal phalanges of the third and fourth digits are lost. There are five digits in the foot.

The fifth anterior dorsal vertebra has a pair of long, curved ribs attached intercentrally. The ribs have the same bicipital appearance as observed in Mr. Daniels' specimen. They are present throughout the dorsal series, apparently missing from the lumbar region, and appearing again in the caudal region. How much of this is due to accident is hard to determine.

The structure of the vertebræ can be partially observed in the specimen. The neural spine was a long, low crest, which ran the entire length of the centrum, with a median elevation, so that on lateral view the spine would be triangular in form. The body of the centrum is expanded laterally into a diapophysis which projects anteriorly. In the posterior region of the dorsal series the mold of the interior of the vertebra shows that the notochord was largely persistent and that the osseous portion of the vertebra was but a thin shell.

The structure of the zygapophyses can not be determined. That they were dorsal in position is, however, evident from several vertebræ. The points of these structures project laterally. There is a notch between the anterior zygapophysis and the roof of the neural canal.

The restoration of the skeleton of this species, given on plate 12, is a summary of existing knowledge of the skeletal anatomy of the genus. Much remains to be determined, such as the arrangement of the scutes of the ventral armature, the anatomy of the pectoral girdle, and the more exact knowledge of the feet and vertebræ. The restoration gives approximately the form of the body and the condition of the skeleton as we know it at present.

MEASUREMENTS OF THE SPECIMEN OF AMPHIBAMUS GRANDICEPS COPE.

	mm.
Length of skeleton	67
Length of skull	15
Posterior width	15
Depth of tympanic notch	4
Width of tympanic notch	6
Long diameter of the orbit	7
Transverse diameter of the orbit	5.5
Interorbital width	4.5
Diameter of the pineal foramen	.75
Length of cervical series of vertebræ	9
Length of dorsal series	35
Length of caudal series	13
Length of centrum of the dorsal series	1.5
Length of a dorsal rib	3.5
Length of arm	20
Length of humerus	7
Length of radius and ulna	4
Width of carpal space	3
Length of third digit	5
Length of leg	25
Length of ilium	3
Length of femur	9
Length of tibia and fibula	5
Length of carpal space	4
Length of first digit	3
Length of second digit	4.5
Length of fourth digit	7
Number of ventral scutellæ in 1 mm	3

The specimen, with obverse, is No. 794 (1234) of the Yale Museum.

Amphibamus thoracatus Moodie, 1911.

(Plate 5, fig. 2.)

The chief diagnostic characters which will at once distinguish the species are: the elongate arm, the large interclavicle, the shape of the vertebra, and the triangular skull.

The portions of the animal which are preserved are: the impression of the skull with one orbit, the right humerus and radius (ulna ?), the interclavicle, the left clavicle, a single vertebral centrum with portions of others, and traces of the scutellæ. These remains are so intermingled with the remains of plants that it has been quite difficult to distinguish bone impression from that of plants. This, however, has been done by whitening the fossils with ammonium chloride, when the texture of the fossils serves to distinguish the one from the other. Parts of the plants have been converted into and destroyed by galena and kaolin, as have also parts of the bones, so that the task has been doubly difficult. There can be no doubt, however, that the observations recorded below are correct. The position of the arm in relation to the pectoral girdle and the position of the girdle in relation to the skull impression first called attention to the possible presence of a fossil amphibian.

There is little to be said of the skull. It is merely an impression in the nodule. It is triangular in form with the snout an acute angle. The angle is, however, exaggerated by the compression to which the fossil has been subjected. The right side of the skull lies over a portion of some plant. The animal is preserved on its back, so that this gives a good opportunity for the study of the pectoral girdle, which is partially preserved. The interclavicle is very large, and from that character the species has been given its name (thoracatus =armed with a breast plate). The interclavicle is an exaggerated "T" with the stem very short. Its anterior margin is curved and ends is a rather sharp, elongate point. The posterior spine is quite short and sharp-pointed, having a length of four millimeters. The interclavicle recalls, in a measure, the same element of *Branchiosaurus,* although it is much more expanded anteriorly and has a shorter spine. In these respects it resembles more nearly a reptilian element. The bone is quite smooth.

The clavicle is of the simple triangular form so characteristic of the Microsauria. It is somewhat displaced backward and its inner margin is slightly obscured.

The humerus is elongate, apparently cylindrical, and has expanded ends. It resembles closely the humerus of *Amphibamus grandiceps,* although its proportions are much greater than in that species. Its length is almost equal to the length of the skull, while in *A. grandiceps* the length of the humerus is only one-half that of the skull.

The radius (ulna ?) resembles in its general proportions those of the humerus. It is a more slender, lighter bone. The impression of the other bone of the fore arm is obscured.

A portion of a single vertebral centrum is preserved. It is from the posterior part of the dorsal series. The centrum is apparently amphicœlous. Its height is about one-half greater than its length. The neural spine is obscured.

The species *Amphibamus thoracatus* Moodie has been described in the Proceedings of the U. S. National Museum, volume 40, page 431, figure 2, 1911.

MEASUREMENTS OF THE TYPE OF AMPHIBAMUS THORACATUS MOODIE.
(Cat. No. 4306, U. S. N. M.)

	mm.
Length of entire specimen as preserved	60
Length of skull impression	18
Greatest width of skull impression	15.5
Long diameter of right orbit	4
Transverse diameter of right orbit	3
Transverse width of interclavicle	14
Long diameter of interclavicle	7
Long diameter of clavicle	9
Greatest transverse diameter	3
Length of humerus	10
Greatest diameter of humerus	4
Least diameter of humerus	1.5
Length or radius (ulna ?)	11
Length of vertebral centrum	2
Width of vertebral centrum	3

Genus CEPHALERPETON, new genus.

This genus is founded on remains of an incomplete individual of a relatively large microsaurian from the Mazon Creek shales. The genus is most immediately related to the family Amphibamidæ, of which two species are known. The present form differs from these species in many respects, notably in size. The skull in the present genus is nearly as long as half the entire body of *Amphibamus grandiceps*, inclusive of the tail. Other structural differences are the anisodont teeth, the large size and more median position of the orbits, and the absence of the posterior tympanic notch in *Cephalerpeton.* The form of the skull recalls that of *Melanerpeton* and *Pelosaurus* of Europe, but they are both branchiosaurians, while the present form, from the structure of the vertebræ and the long, curved ribs, is an undoubted microsaurian. Nothing like it occurs in the Kilkenny, Ireland, fauna described by Huxley, and it is totally different in structure from any of the Linton or Cannelton genera, and its like is not known among the forms from the continent of Europe. It is most nearly approached in certain respects by the various species of *Erpetosaurus,* but from this genus it can be readily distinguished by the smooth skull bones, the absence of a posterior table to the skull, and the presence of a highly developed ventral armature. The interorbital width is less than the transverse diameter of the orbit.

The generic characters are found in the broad skull, the anisodont teeth, the median position of the orbits, the absence of a tympanic notch or posterior table to the skull, the presence of sclerotic plates, the great length of the fore limb and the well-developed ventral armature.

Cephalerpeton ventriarmatum new species.

(Plate 1, fig. 4; plate 7, fig. 2.)

The remains on which the present discussion is based consist of an almost entire skull, twenty-six consecutive vertebræ, both fore limbs, twenty ribs preserved on the right side of the vertebral column, and a portion of the ventral armature.

The skull is very broad posteriorly, its width being one-third greater than its length, with due allowance for crushing. A pineal foramen is not preserved. The sutures bounding the premaxillæ, the maxillæ, the nasals, the prefrontals, the frontals, a portion of the parietals, the squamosal, the supratemporal, the quadratojugal and the quadrate (?) are fairly well preserved. The arrangement of these elements can be discerned by reference to figure 2, plate 7. The prefrontals are unusually large and are triangular in shape. The supratemporal is also quite large. The epiotics and the supraoccipitals are not preserved. The surface of the skull bones is smooth and there is nowhere an indication of sculpture.

Portions of four sclerotic plates are preserved in the right orbit. These measure one-half by three-quarters millimeters. The orbits are large and the interorbital space is less than the transverse diameter of the orbit. Thirteen teeth are preserved on the left maxilla. The teeth are apparently pleurodont. They are short, sharply pointed, smooth and unequal. The first two left maxillary teeth from the anterior end are short. Then follows a tooth which is one-third longer than these two. The fourth tooth is somewhat shorter than the third. The fifth and sixth are still shorter and are practically equal. The seventh, eighth and ninth are all large. The ninth is the largest and the diameter of the base is greater than the third. The last four teeth are practically equal in size, though somewhat larger than the first two.

The right mandible is preserved almost entire, though so badly eroded that little can be said of its structure. Impressions of twelve teeth are present on the mandible, and all are apparently equal. The cotylus seems to have been far posterior and an angle of the mandible projected slightly back of the skull.

There remain only a few indefinite impressions of the cervical vertebræ. The union of the skull with the vertebral column is obscured and lost. Impressions of the dorsal vertebræ are well preserved. Wax molds made from these impressions show the structure of the dorsal vertebræ surprisingly well. The vertebræ are long and cylindrical, with the median portions slightly constricted by a deep pit on each side of the low neural ridge, which takes the form observed in *Thyrsidium, Molgophis, Phlegethontia, Dolichosoma* and other genera. The vertebræ are strongly amphicœlous and the notochord was probably persistent. The sides of the vertebræ are smooth.

The ribs are all intercentral in position, agreeing in this respect with all other Carboniferous Microsauria so far studied. The anterior ribs are very broad near the base and recall the broadly expanded ribs described by Schwarz for *Scincosaurus, Ptyonius, Thyrsidium* and other genera. Posteriorly the ribs become slender and cylindrical. They are all rather long and distinctly curved, with probably a cartilaginous tip.

There is preserved a single element of the right side of the pectoral girdle. This is, I think, the coracoid, an element which has hitherto escaped observation among the American Microsauria. It is long, and spatulate at both ends. Its median portion was apparently almost cylindrical. Its form is not unlike that described by Credner for the coracoid of *Branchiosaurus*, save that the lower end of the branchiosaurian coracoid is acuminate. In the present genus it is spatulate. Its relations with other elements of the pectoral girdle have never been satisfactorily determined.

The fore limbs are both partially preserved. The humerus of the right side is complete. It is greatly elongate for a microsaurian. The form of the element is not unlike that of a lizard. The lower end of the bone is spatulate. Endochondrium seems to have been well developed. Very little difference can be seen between the forms of the arm bones which represent the radius and ulna. They are both elongate, with constricted median portion and expanded truncate ends. The carpus is unossified and the cartilage has left no impression on the stone.

The right hand has two metacarpals preserved, which are fully one-half as long as the radius and ulna. They are separated some little distance from the ends of these elements. This may be due to post-mortem shifting, though the carpus was undoubtedly broad. On the left side are preserved a portion of the humerus, the radius and the ulna, with three metacarpals lying next to the vertebral column. The carpal space is not so large on the left as on the right side. The ventral armature is well preserved in a narrow patch about one inch in length. The chevron-shaped rods are quite large, there being two of them in one millimeter.

The type specimen is No. 796 of Yale University Museum. Collected in 1871, at Mazon Creek, Illinois.

MEASUREMENTS OF THE TYPE OF CEPHALERPETON VENTRIARMATUM MOODIE.

	mm.
Entire length of fossil	98
Length of skull	22
Width across base of skull	28
Long diameter of the eye	10.5
Transverse diameter of the eye	8
Interorbital space	4
Length of mandible	26
Depth of mandible at the coronoidal region	3.5
Depth of dentary	2
Length of a long tooth	2
Diameter of long tooth at base	.5
Length of vertebral column preserved	64
Length of a centrum	3
Median width of centrum	1.5
Length of rib	6.5
Width of rib at base	.33
Length of coracoid	8
Width of coracoid at anterior end	2.5
Length of humerus	18
Width of shaft	1
Distal width of humerus	4
Length of radius or ulna	10.5
Width across proximal ends of ulna and radius	3
Length of carpal space	5
Length of metacarpal	6
Length of ventral armature preserved	24
Number of rods in a length of 5 mm	10

Genus ERPETOBRACHIUM, new genus.

The remains on which the discussion of the present genus is based are contained in a rounded nodule, with obverse, from the Mazon Creek shales, some three and one-half inches in diameter. The matrix is the usual reddish ironstone of the nodules contained in these beds, and the bones have been replaced by kaolin. The parts preserved are the scapula, clavicle, portion of the coracoid, the humerus, the ulna and radius, all of the right side of the body.

The generic characters are apparent in the greatly elongated fore limb, in the exceptionally broad scapula, the long radius and ulna, which almost equal the humerus in length —a character hitherto unknown among Carboniferous Amphibia.

Erpetobrachium mazonensis new species.

(Plate 2, fig. 2; plate 8, fig. 3.)

The scapula of the present form is exceptional in its shape. It resembles an asymmetrical pyramid, the anterior side of the lower edge of the bone being contracted so that the anterior edge is arcuate. Its top is very thin, and possibly terminated in a broad cartilage. The lower end is thick and heavy, and the articular surface is apparently well formed, though somewhat obscured.

The element identified as clavicle is lying on its edge and has the proportions of the clavicle of *Mazonerpeton costatum.* The exterior end is somewhat rounded and small. A portion of another element, which I suppose to represent the coracoid, lies alongside the humerus. Its form is quite obscured.

The humerus has a remarkably well-formed head. Its perfection of formation corresponds well with that of the higher reptiles. Its surface can even be divided into an anterior and posterior articulation. It projects posteriorly for the distance of one millimeter from the surface of the shaft. The shaft immediately below the head is somewhat flattened and has an ovoid section. Further on it becomes more flattened, a part of which is probably due to pressure during fossilization. The distal end is somewhat obscured.

The elements of the fore arm are both preserved, and are approximately equal in size. They are remarkable in tha they exceed or at least equal the humerus in length, although they are not so heavy as that element. They are greatly elongate and slender, with the middle of the shaft only moderately contracted. The articular surfaces are well formed, and both bones were hollow, as was also, apparently, the humerus. The ulna is taken to be represented by the most posterior of the two elements, though the relations of the elements may have been reversed.

The base of the left wing of an orthopterous insect, possibly allied to *Paolia gurleyi* Scudder, lies between the radius and the ulna. The nodule also contains impressions of plants, a portion of a frond of a *Neuropteris*, and the impression of one of the *Cordaites*. Lying next the radius is a slender, elongate element, which may be a rib or a portion of a metacarpal. If a rib it indicates that the animal belongs among the Branchiosauria. The fragment is only one-half as long as the radius and is entirely too obscure to base any conclusions upon. The other characters of the specimen point quite strongly to its microsaurian affinities.

The structure of the articular surfaces of the limb bones alone would indicate the microsaurian relationship of *Erpetobrachium*. It may be provisionally associated in the family Molgophidæ with such forms as *Molgophis brevicostatus* Cope, *Molgophis (Pleuroptyx) clavatus* Cope, and *Molgophis macrurus* Cope, from the Coal Measures of Linton, Ohio.

MEASUREMENTS OF THE TYPE OF ERPETOBRACHIUM MAZONENSIS MOODIE

	mm.
Length of scapula	14
Distal width	6
Proximal diameter	3
Length of clavicle (?)	24
Length of humerus	25
Length of ulna	24
Proximal width	4
Diameter of the shaft	2
Distal width	3
Length of radius	25
Proximal width	4
Diameter of the shaft	3
Width of distal end	4

The type specimen is 799 (222) of Yale University Museum. Collected at Mazon Creek, Grundy county, Illinois.

Genus SPONDYLERPETON, new genus.

The specimen on which the genus is founded consists of nine imperfect vertebræ, from the caudal region, inclosed in a brown ironstone nodule from the Mazon Creek shales.

There have been, up to the present, but two *Carboniferous* genera of the embolomerous Stegocephala described. These are *Cricotus* from Illinois, Kansas and Texas, and *Diplospondylus* from Bohemia. Four, possibly five, species have been assigned to *Cricotus* and a single species to *Diplospondylus* (Fritsch, Fauna der Gaskohle, Bd. 2, Tafln 50, 52, 53). It is with considerable interest that the writer is able to add yet another form to the list of known Embolomeri by the description of the largest form of the Mazon Creek amphibian fauna. The present form exceeds *Diplospondylus* by twice its size and is about two-thirds the size of *Cricotus heteroclitus* Cope.

It differs in several important characters from the two genera above mentioned, but is for the present to be located in the same family, the Cricotidæ of the suborder Embolomeri and the order Temnospondylia.

The present form is distinct generically from any form which have been described. The generic characters are found in the form of the vertebral centrum and in the enlarged intercentra. The present vertebræ are twice as high as wide, differing thus from *Cricotus,* in which the centra are practically circular. A character which is of great importance is the large size of the intercentrum, which almost equals the pleurocentrum in size. It is similar to the pleurocentrum in structure, except for the attached neurocentrum and chevron. The present form differs from *Diplospondylus* in the greater length of the intercentrum and pleurocentrum, in the greater size, in the larger proportions of the neurocentrum and the greater proportionate size of the intercentra.

Spondylerpeton spinatum new species.

(Plate 8, figs. 1 and 2; plate 9, fig. 1.)

The species is very imperfectly known. Sufficient is present, however, to show its wide generic differences from other forms of the Cricotoidæ. These characters are of a phylogenetic nature, and indicate the more primitive nature of the present form, as we would expect from its geological position. The sutures separating the four vertebral elements are clearly apparent. The pleurocentral-neurocentral suture is apparent in four vertebræ.

There is but a single pleurocentrum preserved complete. This shows the form of the attached neurocentrum and chevron, which corresponds to the hypocentrum pleurale, according to Fritsch. These structures are shown in the drawing, figure 1, plate 9. The pleurocentrum is flattened laterally, with a rather large canal for the notochord. Its sides are marked with four longitudinal grooves. Surfaces for the attachment of ribs are not present, and for this reason, as well as the presence of chevrons, the vertebræ are supposed to be caudals. As such, they represent an animal of some three or four feet in length. It was the giant of the Mazon Creek Amphibia.

Attached to the upper side of the pleurocentrum by a sutural union occurs the neurocentrum. The neural arch is quite large and is oval in outline, although somewhat constricted at the tip. The spine of the neurocentrum is rather long and broad at its base, measuring 12 mm. across the anterior zygapophysis. The nerocentrum is laterally flattened and ends in a rather acute and somewhat rugose point. It was probably tipped with cartilage. The anterior zygapophysis occurs well down on the neurocentrum; its lower edge being five millimeters from the suture separating the pleurocentrum and the neurocentrum. The posterior zygapophysis occurs quite high up on the neurocentrum, and lies at a distance of 15 mm. from the pleuro-neurocentral suture, thus indicating an extreme posterior inclination of the neural spine. The posterior zygapophysis of the best-preserved vertebra is separated from its mate, the anterior zygapophysis, on the next succeeding vertebra, by a space of five millimeters.

The ventral surface of the pleurocentrum bears a structure which is without doubt a chevron, although the character of the opening can not be determined. It is elongated and is united by a broad base to the pleurocentrum. Its union is by a clearly defined suture, which is apparent on three vertebræ. The condition represented by the specimen represents almost exactly the condition figured by Cope for the caudal region of *Cricotus crassidiscus* Cope[1].

The intercentrum of the present form is fully as large as the pleurocentrum. The significance of this has already been mentioned. There is no difference, except for the attached neurocentrum and chevron, in the form of the intercentrum and the pleurocentrum. Its body is pierced by the large notochordal canal.

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[1] Cope, E. D., 1890. Trans. Amer. Phil. Soc., vol. xvi, p. 246.

The condition of the vertebral structures represented by the above-described form is so essentially similar to that of *Cricotus*, which has been fully described by Cope, that nothing new can be added to the phylogenetic relations of the separate pieces of the vertebral column. It is remarkable that such a type should be found so low in the geological scale, but it is not unexpected, since we must, without doubt, look to the early Devonian or late Silurian for the earliest types of the Amphibia, which are yet unknown, although they have left their footprints in the rocks of the Devonian and Mississippian epochs of this country and a single skeleton in the Subcarboniferous of Scotland. That our knowledge of the amphibian fauna of the Carboniferous is woefully incomplete is attested by the fact that nearly every specimen collected represents a type distinct from any hitherto known. Such is eminently true of the collection which has just been described. The characters on which the genera and species are based are apparently ones of value, for they have stood the test of time in other groups.

Measurements of the Type of Spondylerpeton spinatum Moodie.
(No. 793 (26) and obverse, Yale University Museum.)

	mm.
Length of specimen	60
Length of pleurocentrum	11.5
Height of pleurocentrum to base of neurocentrum,	20
Length of neurocentrum	33
Width of neurocentrum at base	9
Width across anterior zygapophysis	12
Width across posterior zygapophysis	10
Length of intercentrum	10
Height of intercentrum	10.5
Height of chevron	3
Length of chevron	18
Width of notochordal opening	5
Height of notochordal opening	4.5
Height of neural canal	12
Greatest width of neural canal	6

ACKNOWLEDGMENTS.

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KRAMERIA CANESCENS GRAY.

BY CHARLES M. STERLING.

(Plates 15-22.)

KRAMERIA CANESCENS inhabits the arid regions of southwestern North America. It occurs on the mesa and low hills as scattered individuals; rarely are several of them growing together in groups. The species was reported by Doctor Merriam, in his account of the Death Valley expedition, as common in the dry parts of the valleys of the Muddy and Virgin rivers in southern Nevada. Dr. J. M. Coulter speaks of it, in his Manual of Phanerogams and Pteridophytes of Western Texas, as common in southern and western Texas; and it is said to be particularly abundant along the Rio Grande, where the natives use an infusion of the bark of the roots to dye leather brownish red.

Part of the material for this investigation was collected on Tumamoc Hill, Tucson, Ariz., in July, 1908, by Mr. L. M. Peace; and in October, 1909, The Desert Botanical Laboratory, through the kindness of Doctor MacDougal, furnished additional material for the study of the roots.

The irregular flowers of *Krameria* have made its classification somewhat uncertain. It has often been included in the Polygalæ, but its close relationship to *Cassia* indicates that it should be included in the Leguminosæ or Cæsalpinaceæ. Chodat has placed it in a separate family, the Krameriaceæ, which includes but the single genus *Krameria*, embracing thirteen species, growing in the warmer parts of North America, and in South America as far south as Chile.

Krameria canescens is a low shrub, which attains a height of about 50 cm. The hard, woody stems are usually much branched. The branches may spring from the primitive stem or the bases of other branches which have died down. (Fig. 2, plate I.) During seasons of excessive drouth the stems may be without leaves, but under favorable conditions they are provided with small, narrow leaves. The specimens used for this investigation, collected in the rainy season of 1908, show the stem and branches well supplied with leaves. (Fig. 1, plate I.) Both young stems and leaves are covered by a dense coating of closely appressed trichomes. Many of the branches have been modified as spines, thus reducing the surface for evaporation. The root attains a very great length and thickness in comparison to the low growth of the shrub. This condition corresponds well with the dry climate and scant water supply of the earth in its habitat. The thick main root produces many secondary roots, which have but few branches.

The resemblance of *Krameria canescens* to the South American species of *Krameria,* yielding the official drug, makes it worthy of a somewhat detailed investigation.

A comparison of the tissues of *Krameria canescens* with those of the official South American species shows its close relationship to the species found growing in the northern part of South America and on the adjacent islands. This relationship is also shown by the structure of the flowers in which *Krameria canescens* and those species confined to the northern parts of South America have four stamens and three fully developed petals, while the one species, *Krameria triandra,* growing farther south, has three stamens and two fully developed petals.

THE STEM.

The exceedingly hard, woody stem of *Krameria canescens,* although not presenting any anomalous structures, has several interesting characteristics. The young stems are protected by a cutinized epidermis, which apparently remains functional for a period of three to five yars, and is then replaced by cork. A primary cortex and pericycle, not sharply differentiated, surround the vascular bundles, which form a hollow cylinder enclosing the medulla. (Fig. 3, plate II.)

The epidermis, consisting of a single layer of nearly isodiametric cells, 24 microns in radial diameter, with a cuticle 8 microns in thickness, is in no way unusual (*e,* fig. 3, plate II).

But the dense covering of thick-walled trichomes on its surface provide an excellent means of protection against evaporation. The trichomes are one-celled, with walls strongly cutinized on the lower portions, and are but weakly, or not at all, cutinized on the upper portions. They vary in length from 150 to 600 microns and average 850 on a sq. mm. (*h*, fig. 16, plate IV).

Stomata, which stand at right angles to the long axis of the stem and lie in the same plane with the epidermal cells, appear to be without any of the special protective devices found in desert plants, and are protected only by the numerous closely appressed trichomes (*s*, fig. 16, plate IV). The stomata are uniformly distributed and average 98 on a sq. mm. The cells of the epidermis contain chloroplasts and tannin and have a brownish-red color.

The primary cortex and pericycle are not sharply differentiated, but in the region of the line *d*, fig. 3, plate II, there are numerous parenchyma cells filled with starch. The location of these cells just outside of a zone of tissue containing many bast fibers would indicate that they form the inner boundary of the primary cortex. The primary cortex is made up of a compact palisade and isodiametric or radially elongated parenchyma cells (fig. 15, plate IV). The palisade seldom contains more than one row of cells, which are closely fitted together, and have relatively small intercellular spaces between them. Radially, they measure 30 to 48 mm. in diameter and contain an average of 120 chloroplasts (*cl*, fig. 17, plate IV). The chloroplasts are biconvex and disc-shaped, 4 to 5 microns in diameter and 1 to 1.75 microns in thickness. The rest of the primary cortex is made up on thin-walled parenchyma cells, most of which contain chloroplasts, although many are almost completely filled by large rosette-aggregate crystals of calcium oxalate, while others filled with starch grains are not uncommon. Radially they measure from 25 to 45 microns.

Surrounded by the primary cortex is a pericycle consisting of thin-walled parenchyma cells, and angular, rather strongly lignified bast fibers (fig. 6, plate II). The long, slender bast fibers occur either singly or in groups, and by means of their long, tapering ends are spliced together, forming a tissue well suited for strengthening. They have a few very small straight pits. In length they vary from 850 to 2050 microns and in width from 12 to 18 microns. The parenchyma cells of the pericycle are similar in form and structure to those of the primary cortex. For the most part they contain starch, but in a few may be found chloroplasts, and in many crystals of calcium oxalate. In the palisade and all of the parenchyma cells of both primary cortex and pericycle there are large quantities of tannin, and all have a brownish-red color.

The narrow vascular bundles are collateral, and are separated by narrow medullary rays, one cell in width.

The phloëm is composed of undivided mother cells of sieve tubes and companion cells and thin-walled parenchyma, interspersed with comparatively thick-walled, lignified bast fibers (fig. 10, plate III). The bast fibers are similar in structure and arrangement to those of the pericycle, but a greater proportion of them occur singly, and the groups are smaller. The vertically elongated parenchyma cells contain starch, amorphous proteids, and crystals of calcium oxalate (fig. 11, plate III). Sieve tubes with well-developed sieve plates were not found, but the stems are well supplied with undivided mother cells, which are filled with granular proteid matter, and together with the parenchyma contain tannin in considerable quantities (*n*, fig. 11, plate III).

The elements making up the xylem are fiber tracheids, water tubes, and few wood parenchyma cells. The most conspicuous of these is furnished by the fiber tracheids, which are thick-walled, strongly lignified and compactly arranged, thus leaving very small intercellular spaces. They are cylindrical, and have long, tapering ends, which overlap and are interwoven to make the wood exceedingly strong (fig. 12, plate III). They vary in length from 350 to 1100 microns and 10 to 14 microns in width. Although the thick-walled tracheids are perforated by numerous bordered pores, and are thus well adapted for water conduction, the numerous water tubes are apparently sufficient to perform that function and leave the tracheids to serve rather the function of water storage. (See fig. 8, plate III.) In a stem 4.21 mm. in diameter, the tracheids comprise an area of 5.9 sq. mm., which is equal to 78 per cent of the xylem or 19.7 per cent of the whole stem. The small cavities of the tracheids have a total area of 0.7 sq. mm., which is equal to 8.75 per cent of the xylem or 3.5 per cent of the whole stem.

The tracheal elements are composed of water tubes having spiral and reticulate thickenings and those having bordered pits. Next to the medulla, and formed in the protoxylem, lie the small spiral tubes, while adjacent to them are the somewhat larger reticulate tubes (fig. 9, plate III). But throughout all of the rest of the xylem only tubes with bordered pits are to be found. They have lignified walls 1.5 to 3 microns in thickness, and average about 30 microns in diameter. Although they occupy a much smaller area of the xylem than the tracheids, the water tubes have relatively much greater water-carrying capacity, the total, for the stem given above, being 14.5 per cent of the xylem or 6 per cent of the whole stem.

The tissues of the xylem are not distributed in such a manner as accurately to indicate the periods of growth. However, as the larger water tubes are in rather loosely formed rows, concentrically arranged, and frequently accompanied by rows of wood parenchyma, they appear to form the boundary lines of the periods of growth. The wood parenchyma cells are developed prior to the larger water tubes, thus indicating that the wood parenchyma cells are formed at the end of one growing season and the larger water tubes at the beginning of the season following. The vertically elongated cells contain many circular straight pits (figs. 22 and 23, plate V); and as they are abundantly supplied with starch, they furnish additional evidence that the wood parenchyma is formed at the end of the season's growth for the storage of reserve materials.

Separating the vascular bundles are the numerous narrow medullary rays, usually consisting of a single row of cells. In the xylem portion the walls are relatively thin, lignified in the older part only, and contain numerous straight, circular pits. They are from three to six cells in height, and vertically elongated (figs. 20 and 21, plate V). Frequently the tracheids crowd in upon the medullary ray cells and cause a thickening of their tangential walls (fig. 5, plate II). The cells of the phloëm portions have cellulose walls, and as the outer ones are tangentially stretched the rays become wedge-shaped. In all parts the cells are well supplied with starch—a fact which, taken in connection with their structure, indicates that their chief function is that of storage.

At the center of the stem, surrounded by the vascular bundles, is a rather large medulla. The cells have lignified, pitted walls. Those composing the central part are nearly isodiametric (fig. 25, plate V), while the marginal cells are somewhat narrower and vertically elongated (fig. 9, plate II).

All of these cells are well adapted for storage, and many are packed with well-formed starch grains while others contain large crystals of calcium oxalate. Tannin is abundant throughout the medulla.

THE ROOT.

In making this investigation, roots young enough to show primary structures were not obtainable, and as in all specimens the main root was too large to section and illustrate as a whole, it was necessary to select secondary roots for study. These, however, were found to have their tissues in structure and árrangement almost identical with those of the main root. In transverse section, a root shows a broad bark surrounding a circular xylem (fig. 4, plate II).

The brownish-red cork is developed in regular radial rows, and becomes scaly and dark brown on the exterior (figs. 30 and 35, plate VI). Lying between the phloëm and phellogen is a broad zone of parenchyma cells interspersed with bast fibers (fig. 4, plate II). These fibers are similar to those of the stem, but differ in being shorter, slightly broader, and more irregular in form. They vary from 8 to 20 microns in breadth and from 400 to 1050 microns in length.

The medullary rays and wood parenchyma of the root resemble in every way the corresponding tissues of the stem; but in the root the cells are uniformly larger.

The phloëm corresponds in structure to that of the stems (fig. 30, plate VI). However, there is a difference in the distribution of the bast fibers. In the phloëm of the root the fibers occur in larger groups than in the zone of parenchyma adjacent (fig. 30, plate VI), while in the stem the larger groups of fibers are found outside the phloëm in the pericycle. In the thin-walled parenchyma cells throughout the bark there are large quantities of starch, and crystals of calcium oxalate. The cells have a yellowish-red color, and in all of them tannin is abundant.

The elements composing the root xylem are the same as those found in the stem. The fiber tracheids, varying in length from 200 to 600 microns, are shorter and more irregular in form than are the stem tracheids (figs. 32, 33 and 34, plate VI). Frequently the root trahceids have blunt or irregularly shaped ends (fig. 32, plate VI). The most striking difference between the stem and root is found in the conspicuously large and numerous water tubes of the root xylem (fig. 30, plate VI). Excepting the tubes of the protoxylem, all of the water tubes in roots have bordered pores, which are oblong and lie horizontally in the thick, lignified walls (fig. 31, plate VI). The water tubes vary from 12 to 65 microns in diameter, and have walls from 1.5 to 3 microns in thickness. In a root 4.84 mm. in diameter the area of water-tube cavity amounts to 31 per cent of the xylem or 6.8 per cent of the whole root. The total area of tracheid cavity amounts to 6.7 per cent of the xylem or 1.3 per cent of the whole root. Combining these amounts gives a total for both water tubes and tracheids of 37.7 per cent of the xylem, or 8.1 per cent of the whole root, devoted to the carrying and storage of water. Comparing this data with that given above for a stem 4.21 mm. in diameter, and having a total capacity for both water tubes and tracheids amounting to 23.25 per cent of the xylem, or 8.75 per cent of the whole stem, shows that as a whole the stem has the greater capacity for holding water.

THE LEAF.

The small, narrow, simple leaves are developed in seasons favorable for growth, but in seasons of excessive drought they may be entirely wanting. They are bifacial and have the stomata standing longitudinally at right angles to their long axes (figs. 39 and 40, plate VIII). The stomata, averaging 95 on a square millimeter, lie in the same plane with the cells of the epidermis (fig. 42, plate VIII).

The epidermis, closely resembling that of the stems, bears numerous unicellular trichomes, which give to the leaves a fine silky appearance (fig. 47, plate VIII). The trichomes are usually bent near the base, so that they lie close to the surface and make a thick covering over the entire leaf surface. They vary in length from 150 to 600 microns, and, like the trichomes of the stem, are thick-walled and have the lower portions cutinized, while the upper portions are but weakly, or not at all, cutinized. Although the trichomes on the leaves are very numerous, averaging 450 to 500 on a square millimeter, the number is exceeded by the stems, which bear an average of 850 on a square millimeter.

Lying beneath the epidermis, and extending around the entire leaf, is the palisade tissue, usually consisting of a single layer of cells (fig. 38, plate VII). The cells well supplied with chloroplasts are cylindrical in tangential section, and lie relatively close together (fig. 41, plate VIII). The rest of the mesophyll is made up of parenchyma cells, which, lying close together, thus give but little intercellular space in the leaves (fig. 38, plate VII). No regularly arranged border parenchyma surrounds the vascular bundles.

The very conspicuous water-carrying system is composed entirely of tracheids, which for the most part have bordered pores (fig. 28, plate V); but in the basal portion of the midvein spiral tracheids may be found (fig. 46, plate VIII). The ultimate ends of the veins are composed of large groups of nearly isodiametric tracheids with numerous bordered pores (fig. 27, plate V, and figs. 44 and 45, plate VIII). The enormous number of tracheids furnish adequate capacity for the storage of a large water supply in the leaf. Interspersed with the tracheids are many thin-walled parenchyma cells (fig. 28, plate V, and fig. 43, plate VIII).

The phloëm consists of undivided mother cells, cambiform cells, and parenchyma (fig. 29, plate V). All of these cells have walls of cellulose and are filled with granular proteid matter. The phloëm occupies but a relatively small portion of the bundles, and in no case were well-formed sieve tubes found.

THE CHLOROPLASTS.

Chloroplasts are present in both leaves and stems. In leaves they occur in the epidermis, palisade, and nearly all of the parenchyma of the mesophyll, while in stems they occur in the epidermis, palisade and greater part of the primary cortex, and even in the outer part of the pericycle. Numerically they are about equally distributed in the corresponding tissues of stem and leaf, an average of 128 being present in a leaf palisade cell and about 50 in a parenchyma cell of the stem. In form they are circular and biconvex, varying from 4 to 5 microns in diameter and 1.25 to 1.75 microns in thickness.

STARCH.

Both stems and roots are provided with an abundant supply of starch. The spherical, oblong or ovoid grains are simple, or 2, 3, 4 and 5 compound. In the stem they are located principally in the pericycle, medullary rays, and medulla, but they may also be found in a few of the parenchyma cells of the primary cortex and in the phloëm and wood parenchyma. They vary in size from 4 to 18 microns (fig. 49, plate VIII). Although the root starch closely resembles that of the stem in form and structure, the grains are uniformly larger and more oblong in shape (fig. 48, plate VIII). They range in size from 10 to 40 microns and are very abundant in the parenchyma of all parts of the bark, medullary rays, and wood.

CRYSTALS.

Crystals of calcium oxalate are present in all parts of the plant. They are monoclinic prisms, or rosette-aggregate in form, and vary in size from 2 to 40 microns in diameter. Aggregate crystals 10 to 30 microns in diameter are abundant in the leaf parenchyma, while the small monoclinic prisms, 2 to 10 mm. in diameter, are relatively few in number (*b*, fig. 51, plate VIII). In the stem, rosette-aggregates occur commonly in the parenchyma of the primary cortex and medulla (*a*, figs. 51 and 52, plate VIII), while in the phloëm they are seldom found, but monoclinic prisms are very abundant (fig. 53, plate VIII). The stem crystals are slightly larger than those of the leaf, and in the root they are considerably larger than in any other part of the plant. In the parenchyma cells of the outer root bark large aggregate crystals are found in considerable numbers, and in inner parenchyma, especially those cells adjacent to the bast fibers, are densely packed with monoclinic prisms (fig. 50, plate VIII).

All parts of the plant are well supplied with tannin, and it is especially abundant in the bark of both roots and stems. Excepting the conducting cells of the xylem and phloëm, all parts of the plant are intensely colored, from a reddish brown in the outer cork to a yellowish red in the inner parenchyma of the bark. The color is very persistent, and in order to make a detailed study of the tissues it was necessary to decolorize the material. This could be accomplished only by long bleaching of leaves and root and stem sections in aqueous solution of potassium hydrate and chloral hydrate.

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CONTENTS:

Evidence of Pleistocene Crustal Movements in the Mississippi Valley, ... J. E. Todd.

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EVIDENCE OF PLEISTOCENE CRUSTAL MOVEMENTS IN THE MISSISSIPPI VALLEY.

BY J. E. TODD.

IT has been thought worth while to call attention to the facts bearing on this subject which have been brought out by a recent study of glacial deposits in Kansas. The more novel facts are (*a*) the strong westerly trend of glacial movements during the Kansan epoch, and (*b*) the high altitude attained on the west compared with corresponding levels farther east.

First. The westerly trend is shown (*a*) by the fact that the ice lobe did not reach to the Kansas river in Douglas county, at least not so as to affect its channel, while from near Lecompton to Wamego there is abundant evidence of its filling the preglacial channel and pushing south of it several miles, especially in Shawnee and Wabaunsee counties. The Kansas river was dammed southwest of Wamego, so that its level stood 200 feet above the present stream. This is shown by a boulder-lined outlet connecting with another valley southeast. Similarly a lake was formed in Mission creek valley, west of Topeka, from which boulder-marked channels lead over into the Wakarusa valley on the south.

(*b*) This is shown also by glacial striæ, especially those of higher levels, which record the main movements of the ice sheet. There are also others, particularly those at lower levels, which conform to the direction of the valleys in which they are found, as though formed when the ice sheet was thin and weak. The general movement is shown by the following table of striæ:

Locality, observer, reference.	*Direction.*
Near Omaha, six or eight feet above the river C. A. White—Geol. Ia., vol. I, p. 95.	S. 51° W.
Near mouth of Platte river Meek—Final Rept. Neb., p. 92.	S. 20° W.
Two miles east of Pacific Junction, Iowa J. A. Udden—Geol. Ia., vol. 13, p. 177.	S. 12° and 14° W.
Three miles south of Pacific Junction, Iowa J. A. Udden—Geol. Ia., vol. 13, p. 177.	S. 25°, 29° and 34° W.
Three miles east by north from Tabor, Iowa J. A. Udden—Geol. Ia., vol. 13, p. 177.	SW.
One-half mile south of Hinton, Iowa J. A. Udden—Geol. Ia., vol. 13, p. 177.	S. 7° and 50° W.
One mile north of Macedonia, Iowa J. A. Udden—Geol. Ia., vol. 11, p. 269.	S. 2° W. and S. 10° E.
Three to four miles south of Pacific Junction, Iowa J. E. Todd—Bull. U. S. G. S. 158, p. 69.	S. 9°, 40° E.
South of Plattsmouth, Neb., 40 ft. above river J. E. Todd—Bull. U. S. G. S. 158, p. 69.	S.-SW.
Bennett, Neb. J. E. Todd—Bull. U. S. G. S. 158, p. 69.	S. 17°, 41°, 61°, 88° W.
Weeping Water, Neb. E. H. Barbour—Neb. Geol. Surv., vol. 1, p. 169.	S. 29° W., S. 11° E.
One mile north of St. Joseph, Mo., 125 ft. above river J. E. Todd—Geol. Rept. Mo., vol. 10, p. 121.	S. 26° W.
One mile northeast of Kansas City, 130 ft. above river J. E. Todd—Geol. Rept. Mo., vol. 10, p. 121.	S. 7°, 9°, 24° W.
East part Kansas City, 100 ft. J. E. Todd—Geol. Rept. Mo., vol. 10, p. 122.	S. and S. 6° E.
Seneca, Kan. L. C. Wooster—Amer. Geologist, vol. 10, p. 131.	S. 21° 24° W.

(*c*) The ice sheet entering Kansas was from the Des Moines valley rather than from the James and Missouri valleys. This is shown clearly by the distribution of red quartzite boulders, for they are found very abundantly on the extreme western edge of the ice lobe. If the ice had come down the Dakota-Missouri valley they would have been distributed only to the eastern half of it, and therefore would have been far east of the marginal effects of the ice, for the original ledges extend only a short distance into South Dakota.

Moreover, the red boulders frequently abound in white and red pebbles, such as are not known to occur in the quartzite ledges of South Dakota, but are found farther east in Cottonwood and Rock counties, Minnesota, where the original deposits were nearer the old Archæan shore on the northeast, which furnished the material. There seems no doubt, therefore, that the ice of Kansas in the Kansan epoch passed through the upper part of the main Des Moines river valley.

This conclusion discounts strongly an oft-used map of North America, professing to show the ice streams at the maximum extent of the ice, and we may the more easily admit its inaccuracy when we know the impossibility of a slender ice lobe maintaining itself for 500 miles from Dakota to Kansas along the edge of a dry and probably warm region like the great plains, doubtless traversed then as now by southwest winds.

Second. Another significant fact is that the ice overrode points now over 1500 feet A. T. This was true in northern Pottawatomie county, about Blaine and Wheaton, Kan., where the preglacial surface rises to that height. The surface of drift deposits lies at that altitude at Summerfield in Marshall county, Kansas, and along the divide northward past Virginia and other points in northern Gage county, Nebraska.

This fact should also be contrasted with the fact that the highest points of the limit of the ice on the east side of the same lobe in northeastern Iowa is only about 1200 feet A. T. in Winneshiek, Allamakee and Dubuque counties. It should also be compared with the fact that southeastern Iowa now lies only 750 feet A. T., or only half the height in Kansas.

Taking the altitudes as we now find them, we can not see why, if the ice sheet reached 1500 feet in Pottawatomie county, Kansas, it should not have pushed over the 1300-foot levels in northeastern Iowa and scores of miles farther into the Wisconsin driftless area, and well across Illinois southeast, in which direction there was an open field and lower levels. We may reason it thus: Taking a point in northern Kossuth county, Iowa, as a common point of passage for all points south from Kansas to Illinois, then taking Blaine, Kan., 1500 feet A. T., and assuming an average slope for the surface of ice of 25 feet per mile, we should find the top of the ice sheet over northern Kossuth county to be over 9000 feet A. T. If a similar slope prevailed also southeast to West Union, Iowa, the ice would have reached 5275, or 4000 feet above the height of the present surface there, and in a southeast direction the slope would have carried it beyond Bloomington, Ill. This is also far beyond the observed limit in that direction.

Mr. J. E. Carman, in the Illinois Geological Survey Bulletin 13, represents the limit of the Kansan till reaching nearly to Savannah, Ill., and a little beyond Fulton, Ill.; and Leverett, in the U. S. G. S. Monograph XXXVIII, places the margin through Hancock and Adams counties, and crossing the Mississippi river near Hannibal, Mo. With the slope assumed and the surface as now, the ice should have gone 70 miles farther in that direction.

Now, if ice may be used roughly in this way as a level, as seems reasonable, it affords evidence that since the Kansan epoch there has been a sinking of the Mississippi region, or a rise of the Kansan, or both. That the first is in a measure true seems attested by the fact that the trough of the Mississippi from St. Paul to Quincy, Ill., is 100 to 150 feet deeper than is demanded by the level of the present stream. Professor Calvin called attention to this fact in his paper in the Proceedings of the Iowa Academy of Sciences, volume 14, page 213. Bedrock is from 150 to 220 feet below low water when the depth adequate for the present stream is only 50 or 60 feet.

Another evidence of the same movement is found in the strong easterly trend of the Iowan ice sheet in eastern Iowa as compared with that of the Kansan there.

This, however, is not enough to fully explain the facts. That there has also been a rise of the Kansan side seems probable, not locally, but in the general westward elevation of the plains. Formerly, when the deposits of the plains were thought to be of lacustrine origin, it was common to speak of the Pleistocene elevation of the Rocky Mountains. The Fluviatile theory has relieved the necessity for that view; but may there not have been some movement of that sort?

An argument in favor of this is found in the fact that quite generally along the Kansas streams and the Missouri river in this latitude there is abundant evidence that the drainage was 85 to 100 feet higher than now. The preglacial channel of the Kansas river at Manhattan was about 100 feet higher than that of the present channel. It was a little lower at Topeka, while at Lawrence a terrace of later date is well developed, with numerous boulders at the bottom of alluvium, and its top 80 to 100 feet above the present stream. Glacial striæ around Kansas City are not found below 125 feet above the Missouri, although there are numerous ones above that level. At Weston, Mo., a cobblestone stratum twenty feet thick, containing red quartzite and granite boulders, is found about 150 feet above the present level. (See Missouri Geological Survey, vol. 10, p. 146.) This terrace, due to the recent cutting down of the Missouri river, has been ascribed to the lowering of the channel into the Ozark limestones in central Missouri, but quite as plausibly it may be ascribed to a Post-Kansan elevation of the region. It need hardly be stated that along the Missouri through the Carboniferous rocks, bedrock is rarely more than 50 or 60 feet below low water. In some shaly portions and in the soft Cretaceous sands and clays of the north it is 100 to 125 feet below.

We can conclude, therefore, that the crust was raised at least 100 feet in Kansas, and depressed 100 feet in eastern Iowa and Wisconsin, or a total relative movement of 200 to 600 feet of eastern subsidence since the time of the Kansan ice sheet. May not the less easterly movement of the ice of that time be further accounted for by a more rapid rise of the underlying rock surface on the east and the southeast? May not the southwesterly trend of the low ledges traversing the eastern part of Kansas, with the corresponding direction of the tributaries of eastern streams, have had an appreciable effect, favoring the westerly movement of the ice in that region? May not the greater heating of the western half of the ice lobe, whether from the maximum daily water temperature coming in the afternoon (see Science, new series, vol. 14, pp. 749-1901), or from the warm southwesterly winds, have rendered the west half of the ice sheet more active, and so increased its westerly movement.

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CONTENTS:

THE ACTION OF SALT SOLUTIONS ON STRIPS OF THE FROG'S INTESTINE ....*Grace Russell.*

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THE ACTION OF SALT SOLUTIONS ON STRIPS OF THE FROG'S INTESTINE.

BY GRACE RUSSELL.

(Plate 23.)

(From the Physiological Laboratory of the University of Kansas.)

IN this paper are briefly described the effects of salt solutions of various concentrations upon the smooth muscle fibers, especially the longitudinal, in the intestine of the frog.

The effect of salt solutions upon smooth muscle fibers has been studied by many investigators, but the works of McGill,[1] Mathison,[2] Meigs,[3] Stiles,[4] Langley,[5] Menis[6] and Row[7] I believe correspond more closely to my problem than do the works of others that I read.

The method employed was very simple. After the spring frog was pithed, the abdomen was opened, and pieces about one centimeter long were removed from the intestines as required. The segment was flushed with Ringer's solution and attached to a writing lever at one end, properly weighted to secure tonicity of the intestinal strip, and suspended in a glass cylinder, into which the solutions to be tested were carefully placed. The lower end of the segment was attached to a small siphon tube, which drained the fluid.

In each case a record was first secured with the strip in isotonic Ringer solution. This was then siphoned off and replaced by the special salt which was to be studied. In no case was the same strip used twice. It was interesting to note that strips from some frogs when suspended showed peristaltic action, while those from others did not. But as soon as the salts were added the peristaltic movements either decreased or ceased. The period of action of each solution was limited to about eight minutes and was followed in each instance by an isotonic Ringer solution. I did not attempt to determine whether the observed effect of the salt was upon the muscle, contractile tissue or nerve sells or nerve endings.

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[2] Mathison, G. C., 1911, Journal of Physiology, vol. XLII.
[3] Meigs, E. B., American Journal of Physiology, vol. XXII, 1908.
[4] Stiles, P. G., 1903, American Journal of Physiology, vol. VIII, No. 4, p. 269.
[5] Langley, 1911, Journal of Physiology, vol. XLII, Proceedings, p. XXIV.
[6] Menis, 1911, Journal of Physiology, vol. XLII, p. 326.
[7] Row, R., 1904, Journal of Physiology, vol. XXX.

After many trials an isotonic indifferent Ringer solution was secured, in which the moist strip neither relaxed nor contracted, but continued its peristaltic movements. The following solutions were then employed in the experiments: the influence of, first, slight alkalinity; second, slight acidity of Ringer solutions; third, NaCl in strengths from m/32 to m/8; fourth, KCl from m/64 to m/8; fifth, $CaCl_2$ from m/64 to m/8; sixth, $MgSO_4$ from m/32 to m/8; seventh, $BaCl_2$ from m/64 to m/8. Also, the influence of tap water and double-distilled water were determined.

I might state at once that tap water caused a marked contraction, possibly due to the large percentage of calcium, while double-distilled water had an opposite effect, namely, that of relaxation, probably due to the extraction of salts from the tissues. This corresponds with Meigs' and McGill's results on smooth muscles from the stomach of the frog with hypotonic solutions.

The accompanying plate illustrates characteristic curves from the different solutions and their most pronounced effects.

A neutral Ringer solution was made .04 per cent alkaline by adding Na_2Co_3 to it. In this solution the peristalsis ceased at once, and the contraction reached its height within a half minute. Though the per cent of alkalinity was small, the effect was quite marked, producing contractions in each case.

In a .04 per cent acid Ringer solution, made so by adding HCl, peristalsis ceased at once, and within a half minute reached its limit of relaxation; relaxing seldom to the same extent as the alkaline Ringer contracted. An m/64 to m/32 NaCl produces a prolonged relaxation, whereas an m/32, and in some cases m/8 NaCl, proved indifferent.

In an m/32 and m/16 KCl solution the contraction of the specimen was immediate and rapid and continued for about four minutes, when it gradually relaxed. But it never returned to normal. In m/64 KCl it was often indifferent, and in m/32 KCl slight relaxation took place. An m/8 KCl strength proved toxic.

It is interesting to note that in $CaCl_2$ solutions, after peristalsis ceased the contractions usually began at once, lasting from one to five minutes, and then in most cases the strip gradually returned to its original length. This tendency of contraction and then returning to its original length was most pronounced in m/32 solutions. The above results with NaCl, KCl and $CaCl_2$ agree with those obtained by Stiles.

In $MgSO_4$ solutions, ranging from m/8 to m/32, intestinal strips relaxed, but relaxation was often more pronounced in an m/32 solution. Magnus in one of his papers states that $BaCl_2$ was a strong stimulus to the intestines and always causes a contraction. My results corroborate his statement, as is shown by the curves. The specimens did not reach their original length after the first contraction, and in each case the contractions were very pronounced.

This work was pursued under the guidance of Dr. I. H. Hyde, to whom I am under great obligations for help and advice.

The general conclusion drawn from the study of the above experiments is that the contraction of the intestinal strip of the frog is produced by solutions of $BaCl_2$ from m/64 to m/8; KCl from m/32 to m/16; $CaCl_2$ from m/64 to m/8 and alkaline Ringer solutions; while acid Ringer, NaCl from m/32 to m/8 and $MgSO_4$ from m/32 to m/8, cause relaxation. These results are probably due to the action of the solution on the longitudinal muscle fibers in the frog's intestine.

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CONTENTS:

On a Comparison of Three Skulls: Castoroides ohioensis, Castoroides kansensis, and Castor fiber.......... *H. T. Martin.*

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ON A COMPARISON OF THREE SKULLS: *CASTOROIDES OHIOENSIS, CASTOROIDES KANSENSIS,* AND *CASTOR.*

BY H. T. MARTIN.

(Plates 24-27.)

(Contribution from the Zoölogical Laboratory, No. 199.)

SINCE the first discovery of *Castoroides ohioensis* and its description, in 1838, by Foster, there has been approximately twenty specimens, mostly fragmentary, reported from as many different localities. Out of this number it is interesting to note that only one, that from Dallas, Tex., was found west of the Mississippi valley. This, coupled with the fact that the Boicourt specimen figured in this paper is a new addition to the Pleistocene of Kansas, adds much to the value of the specimen.

Through the courtesy of the secretary of the Smithsonian Institute, the writer has been allowed the privilege of figuring and comparing with our Kansas specimen a nearly complete skull of *Castoroides ohioensis.* So perfect is this specimen that only a small portion of both the malar arches is all that is missing. The skull is beautifully preserved, with the dentition absolutely perfect. The double posterior nares, well shown, is formed by the pterygoides being laterally compressed at about their middle, until they meet, thus forming two orifices instead of one, as in all other rodents. The superior fossa, pyriform in shape, the lower and smaller one triangular. (See plate 25, fig. *A*, at *a-a*.)

It is unfortunate that so little is known of the history relating to the discovery of the beautiful skull which the writer has been very kindly allowed to figure along with the Kansas specimen. The only data that seems to be attached to the specimen is that it was discovered in Lenawee county, Michigan, and presented several years ago to the Smithsonian Institute by Doctor Kost, and now constitutes No. 1634 of the Smithsonian collection.

Of the two skulls figured in this paper, the Smithsonian specimen from Lenawee county, Michigan, is by far the more complete. This and the skull described by Hall and Wyman, and known as the Clyde skull, are the two most perfect specimens known. The specimen found at Boicourt, Kan., although imperfect, still retains enough of the elements of the skull to warrant the restoration shown on plate 24, figure 8.

After a careful comparison between the Lenawee specimen, and with descriptions of other specimens, several differences occur in the first-mentioned skull which can scarcely be attributed to either age, sex or individual morphological differences. These variations, in the writer's estimation, should be considered enough of a specific character to determine it a different species from *C. ohioensis;* hence the name *C. kansensis* is proposed.

As a comparison, the well-known skull of *Castor fiber* has been used.

HISTORY OF THE BOICOURT SPECIMEN.

The skull figured on plate 26, figure *B*, was donated to the University of Kansas Museum about a year ago by Dr. J. R. McLeland, of Pleasanton, Kan. All that can be gathered relating to the history of the specimen is that about twelve years ago, while sinking coal shaft No. 2, three and a half miles southwest of Biocourt, Linn county, Kansas, in the valley of the Marais des Cygnes, a miner, Mr. W. J. Thirwell, came across the skull at a depth of thirty-four feet, in a layer of sedimentary material of a bluish color, which overlay a deposit of sandy conglomerate. At the time of the find the large incisor was complete, and other parts of the skull were present. For several years this fine specimen was kept in a cigar store, in a case along with the cigar boxes, unnoticed by anyone, until a year or two ago Doctor McLeland recognized in it a fossil form, secured it, and presented it to the University.

In the vicinity of Boicourt, and for several miles above and below, the river Marais des Cygnes has cut down to an average depth of from thirty to forty feet, and in many places the Bethany Falls limestone is exposed in considerable areas at the bottom of the river. About two and a half miles below where the specimen was exhumed, and just below the bridge that crosses the Marais des Cygnes at Trading Post, a small deposit of bones was found several years ago by Mr. Amos Tubbs, of Trading Post. This small collection the writer examined a year ago, and recognized in it teeth of elephant, horse, camel, together with limb bones and a tusk of an elephant, all belonging clearly to the Pleistocene. At the point where the bones were found, the river has cut clear down to the Bethany Falls limestone. Directly above the limestone occurs a layer of conglomerate about eighteen inches thick, above this a like thickness of bluish-gray silty deposit in which the bones were found. The depth at which these were discovered and the material in which they appeared tallies well with the data at hand concerning the *Castoroides* skull, so that it will not be unreasonable to suppose that at one time the layer in which the skull was found was once the old river bed, or the bottom of some body of water adjacent to the river. The fact that the bones found at Trading Post were in the same deposit and at about the same depth clearly indicates that *Castoroides* was contemporary with these forms.

To those unacquainted with the geology in the vicinity of Trading Post it may be well briefly to enumerate the layers as they occur where Mr. Tubbs' bone bed was exposed:

From the general level of the valley:
- Six feet of black loam.
- Twelve feet marly clay.
- Twelve feet blue and yellow marl, verging into shale.
- Eighteen inches of bluish silt in which the bones were found.
- Eighteen inches conglomerate lying on the heavy Bethany Falls limestone.

The conglomerate beds and the blue silty material occur only in isolated patches of a few feet in length; that in which the bones were found was probably twenty-five feet in length.

THE SKULL.

The typically rodent-like skull more closely resembles that of *Castor fiber* than of any other of the rodent family, yet in many respects close analogies are found to that of the Bizacacha (*Lagostomus trichodactulus*), a living form found on the high pampas of South America. In tooth structure it more closely resembles this form than any other of the rodent family I have been able to examine, while the long diastema between the incisors and the molars, and rapid divergence of the molar series posteriorly and the general form of the basioccipital region also appear more Bizacacha-like. Although the molar teeth of these two forms look very similar on their grinding surface, there is quite a difference in their conformation. Those of the Bizacacha are made up of two laminæ of dentine and enamel in molars 1, 2, and 3, the fourth and last having three layers. The enamel does not form a true cylinder around the dentine, but only reaches part way around, there being no enamel wall on the posterior portion of the segments. In *Castoroides* each tooth has one more layer, a tooth being made up of tube-like sections composed of enamel and dentine, pressed nearly flat and fastened together with a layer of cement.

The dorsal surface of the skull is almost flat, broadening out posteriorly, and in the region of the lamboidal ridge is relatively more broad and massive than in *Castor*.

The narrowest part of the skull occurs just behind the cephalic ends of the parietals, instead of across the frontals as in *Castor*. Across the frontonasal sutures it presents a more massive appearance, and from here it tapers slightly to the tip of the nasals, which end rather abruptly and rounding. The facial portion of the maxilla composes a larger part of the zygomatic arch than does that element in *Castor*, and forms the major part of the front wall of the orbit. In consequence of this, the malar does not reach nearly so far forward, and has no contact with the lachrymal, while the contrary exists in *Castor*, where the front portion of the malar forms half of the anterior wall of the orbit. The squamosal extends further forward and commences higher up on the cranial portion than it does in *Castor*, and occupies about two-fifths the entire length of the skull, while the nasals are relatively more broad than long. The long and narrow parietal ridge rises sharply from the flat parietals to a height of ten millimeters. The interparietal is a relatively small element, almost unobservable, fitted wedge-like between the posterior ends of the parietals. The parietals themselves are long and very narrow, scarcely exceeding in width that of a full-grown beaver. The infraorbital foramen is located higher up the face than in *Castor*.

The diastema between the molar series and the massive incisor tooth differs much comparatively in the Lenawee and Boicourt specimens. In the former it measures 110 mm., in the latter 140 mm., and is, proportionately to the size of the skull, much longer.

A comparison of the large incisors of the two specimens reveals quite a difference in the markings of the outer enamel covering. In the Boicourt specimen the flutings are very narrow, and, counted from the anterior inner angle round to the posterior sinus, there are twenty-four. Correspondingly, the Lenawee skull shows only eighteen, while two of the fluted groovings on the anterior face of the teeth are exceedingly wide, the widest being 8 mm. from ridge to ridge, and widens out materially as it nears the extremity of the tooth.

In *Castor* the cutting edge of the incisors is worn nearly straight across, with slightly more wear on the outer edges, but in the *Castoroides* quite the reverse occurs, the teeth being worn thin and sharp at their junction on the medial line, and gradually get thicker and more heavy laterally, the two teeth thus forming a very efficient gouge.

A lateral view of *Castoroides,* when compared with *Castor,* shows quite a difference in contour of the dorsal part of the skull. In *Castor* the crest from supraoccipital to nasals shows quite a convexity, the highest part of which occurs at the junction of the parietals and frontals on the median line, and more strongly convexed at the caudal end of the nasals than at any other point, with the parietal ridge scarcely noticeable. In *Castoroides* the parietal and saggital ridge is very conspicuous, and is elevated 10 mm. above the parietals. From the supraoccipital ridge to the frontonasal suture on the median line it is nearly level; forward from here to the tips of the nasals it has a gentle slope. The malar is wider, thicker and more rounded than in *Castor,* with the lower portion gently converging inward. The cephalic edge, from the superior malomaxillary suture downwards, has a slightly more acute angle than has *Castor.* In *Castor* the malar on its outer face is nearly flat its whole depth from top to bottom. The squamosal is relatively much larger than in *Castor,* and extends higher up the sides of the cranium, thereby decreasing the width of the parietals. The infraorbital foramen in

Castoroides, compared with *Castor,* is decidedly larger, a third deeper than wide, and occupies a more dorsal position proportionally, having a long, deep fossa extending cephalad and dorsad, which is wider and shallower anteriorly. In *Castoroides* the superior portion of the maxilla in the orbitosphenoidal region is smoother, and does not have the excrescences caused by the roots of the molars as in *Castor.* With the exception of a few of the foramina of *Castoroides,* all obtain a proportional enlargement over *Castor.* One very noticeable exception is the small size of the external auditory meatus. In this there is but a slight difference in the actual size between the two, and is not nearly as large as that organ in Bizacacha. The tympanic bullæ in *Castoroides* are very small, and but little inflated, while the basioccipital is comparatively much broader and shorter. The double posterior nares, and the peculiarly constructed pterygoides, that have attracted the attention of all who have made a study of *Castoroides,* are beautifully shown in the Lenawee specimen belonging to the Smithsonian Institute. (See plate 25, fig. *A,* at *a-b.*) The mastoid in *Castoroides* is an exceedingly stout and massive bone, compared with *Castor;* the outer edge of this bone and the sternomastoid are deeply scored and pitted for heavy muscular attachment. The glenoid fossæ in *Castoroides* are shallow and broad, and would allow a more lateral motion of the lower jaws than would those of *Castor.* At the base of the auditory bullæ, and in front of the mastoid of *Castoroides,* occurs a deep and rough pit for muscular attachment that is not found in *Castor.*

On plate 27, figure *B,* a lateral view of the Boicourt specimen is given. The large broken incisor is withdrawn from its socket and the thin inner wall removed, showing the depth of the socket, which reaches back to the roots of the first molar, a thin partition only separating the pulpy root of the incisor from that of the molar. Just above, at *b* and *c,* are seen the frontal and nasal sinuses. A portion of the inner wall of the maxilla has been removed to show the last molar (*d*) as it lies in its alveolus, while at *e* is shown the brain cavity.

In Castoroides kansensis.

The basic occipital shows no sutural contact with the basic sphenoid, but is firmly coössified with that element, and while the conical projections mentioned by Wyman exist, they are situated more caudally and closer to the occipital condyle than in either the Lenawee specimen or that described by Wyman from Clyde.

The pterygoid, or wall between the internal pterygoid fossa and the inferior entrance to the posterior nares, rises immediately from the projection mentioned by Wyman, but at a point much farther back than in his specimen or that from Lenawee; consequently the pterygoids would be proportionally much longer transversely, while just behind and above the foramen rotundum two cavities, one above the other, occur, extending forward and upward, with the evidences of having a thin septum between the outer and inner walls of the internal pterygoid fossa.

Probably one of the most striking features of *Castoroides* is its enormous incisor teeth, which appear much out of proportion to the size of the molar teeth and skull.

The principal differences observed between the Lenawee and the Boicourt skulls, then, are: The proportionally larger incisors, with narrower grooves and greater number; the relatively longer diastema between the incisors and molars; the difference in the pattern of the folds in the last molar, and its comparative smallness; the difference in frontonasal region; comparatively longer nasals; and deep, rough pittings of the parietals.

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MEASUREMENTS OF THE TWO SKULLS.

	Lenawee county, Michigan, skull. *mm.*	*Boicourt, Linn county, skull.* *mm.*
Greatest length from tip of nasals to occipital condyle...	270	....
Nasals, greatest length..............................	93	110
Nasals, greatest width..............................	53	62
Frontals, greatest length..............................	84	..
Frontals, width at postorbito-frontal suture...........	84	
Parietal, length	133	....
Parietal, width at point opposite squamosal...........	53	61
Molars, length of series............................	68	76
Grinding surface of first molar, length...............	17	20
Grinding surfaces of first molar, width...............	16	19
Grinding surface of last molar......................	20	20
Incisor, lateral diameter at level of alveolus...........	24	28
Incisor, transverse diameter.........................	21	25
Occiput, transverse diameter.........................	149	
Occiput, vertical diameter...........................	67	....
Depth of skull, level of molars to frontal bosses........	110	120
Length of diastema between incisor and first molar.....	110	140

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II. Cytological Studies of Femur-rubrum and Other Melanopli.
Nadine Nowlin.

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CYTOLOGICAL STUDIES OF *FEMUR-RUBRUM* AND OTHER *MELANOPLI.*

BY NADINE NOWLIN.

(Plates 28-32.)

(Contribution from the Zoölogical Laboratory, No. 200.)

FIFTEEN years ago Wilcox ('95) published a paper on the spermatogenesis of *Melanoplus femur-rubrum*, or *Calopterus femur-rubrum*, as the genus was then called. It is generally conceded that many of his interpretations are incorrect, but up to the present time his work stands as the only description of this species. Since it is my purpose to examine the chromosome complexes of as many genera of *Melanoplus* as possible, it seemed advisable to give emphasis to the discussion of this particular one.

Of the some dozen species that I have examined, *femur-rubrum* is perhaps the least favorable for study. The stage in which the chromosomes separate well are very short, and counts are difficult. This may explain some of the errors that Wilcox made.

MATERIAL AND METHODS.

The grasshoppers were collected in September, 1910, on the south campus of the University of Kansas. This is not the best time for collecting, as the germ cells are nearing the end of their activity, but all stages can be found from spermatogonia to spermatozoa. Wilcox's material was collected also in the fall. I have used Flemming's and Bouin's fluids for preserving, and both gave good results. Heidenhain's iron-hæmatoxylin was the chief stain, but Flemming's tricolor was also tried.

DESCRIPTION.

Melanoplus femur-rubrum.

Spermatogonia.

It was somewhat difficult to find clear spermatagonial plates, as the chromosomes crowd very closely at this phase, and the bent form of the rodlike chromosomes seldom lie in one plane. Their general arrangement is radial, one end of the rod pointing toward the center of the cell. Figure 5 is an unusually fortunate cut, which shows the radial arrangement very well. Many of the chromosomes are in cross sections, though two or three pairs of small spherical ones occur rather constantly through the later generations. The chromosomes can be paired fairly accurately in such a view as figure 5.

Going back to an earlier stage in the development of such a cell, we find the chromatin in the form of a slender spireme, radiating from a black mass lying close to the nuclear membrane. (Fig. 2, pl. 28.) This body is probably the accessory chromosome. The spireme is irregular, giving a beaded appearance. I take figure 3 to represent the next stage. It lies in an adjoining cyst. The thread has thickened, segmented across, and shows a longitudinal split. At figure 4 is shown a still further condensation of chromatin. Chromosomes are now formed which look vacuolated, and one is distinctly separated from the others by a membrane. Such an isolation of the accessory is typical of *Brachystola* and other genera. The metaphase, viewed from the side of the spindle, looks somewhat like a second spermatocyte, but here, of course, the chromosomes are more numerous and crowded, and the spindles smaller. An equatorial plate has been described at the beginning of this section. It shows twenty-three chromosomes when all are present, and in many of the cells it is not difficult to identify the chromosomes of spermatognia and spermatocytes.

Oögonia and Female Somatic Cells.

A study of developing oögonia was made in the hope of getting clearer plates than in the male, but the cells are very small when division takes place. After division rapid growth occurs, in which the chromatin forms a fine spireme, much beaded in appearance. (Fig. 12.) The nucleus grows as well as the cytoplasm, the fine chromatin threads expanding into broad, faintly staining bands. (Fig. 13.) Finally the chromatin forms faint rings (figure 14), and the oögonia have all the appearance of cells in prophase, but instead of continuing division, now there follows a long rest and further growth stage, in which the nucleus takes the typical germinal-vesicle form, and the cytoplasm fills with yolk.

Many dividing somatic cells are to be found around the growing eggs in the follicular tissue. Numerous attempts to count the chromosomes were made, and the greatest number found was twenty-two. I take this to be incorrect, however, as counts on other female cells have given twenty-four. In *Stenobotherus* (McClung 11) there are twenty-two chromosomes in the female, but two of these are multiples. I have not been able to identify any multiple in *femur-rubrum*.

Wilcox says, on page 9 ('95): "I could not determine how many divisions the spermatogonia undergo. The chromosomes in the prophases are twelve in number, twenty-four at the equator of the spindle, during metakinesis. The individual chromosomes are rod shape, or often elongate spindle shape. In metakinesis they show ordinarily the well-known V-shaped figures, and are connected with each other in pairs by means of linin fibers." Whether this writer had confused second spermatocyte and spermatogonia I do not know. At any rate, his count for spermatogonia is incorrect. And it is even more peculiar that he considered the divided twelve as twenty-four chromosomes instead of two newly formed cells, each with twelve chromosomes.

Spermatocytes.

The early spermatocytes show faintly staining nuclei, with chromatin scattered in loose threads of varying lengths. The threads seem irregular in diameter, giving the effect of a greater amassing of granules at certain points. (Fig. 16.) The threads assume more definite outlines later and become finer. They still have the beaded appearance, and at one time form a bouquet stage, all looping out of a darkly staining chromatin mass at the periphery of the nucleus. (Fig. 17.) The beaded appearance is seen just before the spireme breaks up (figs. 19, 20), and even in the early chromosomes this appearance is retained. Wilcox believed that four of these nodules are grouped together to form the tetrad, and that what we know as the tetrad is made up of four chromosomes. He believes the first spermatocyte division effects the separation of two of the chromosomes from the other two of the original quadrivalent group. And in the second division the remaining chromosomes that have hung together in pairs are parted.

I believe his whole error has come about through the beaded appearance of the spireme. It is true that there are distinct nodules in this thread, and that in the formation of the chromosomes one sees often four knots to a single thread, but one sees many more sometimes. Instead of considering them individual chromosomes, we must believe that they are mere irregularities in the amassing of granules during spireme formation.

We get some cells which show a very clear case of a continuous spireme. (Fig. 18.) The chromosomes in the early prophases are in the form of rings and crescents. The very early rings show sometimes the longitudinal split, but it is seldom. The first tendency in the condensing of the chromosomes is to form rings, thus in figure 2, plate 29 many are seen. When they begin to arrange for division we see fewer, usually from two to three. Some are crosses, and even in the prophase have begun dividing. (Fig. 3.) There are twelve chromosomes, then, in the first spermatocytes, formed, as I believe, by the union of the pairs of spermatogonial chromosomes; one of these is the accessory and is unpaired. (Fig. 16.) This special chromosome moves toward the pole ahead of the others undivided (figs. 1, 2, 4, pl. 30), and in equatorial plate is seen always in a different plane from the rest. The remaining eleven divide longitudinally (figs. 1, 2, 3, 4), so that the number of chromosomes distributed to the daughter cells is eleven and twelve respectively. We see the two sets of plates, then, in the second spermatocytes. (Figs. 6, 7.) By means of a cross division of these chromosomes the eleven diads are separated. The accessory divides longitudinally. Thus we see that half the spermatids have eleven chromosomes plus the accessory (fig. 6), and half have the eleven and no accessory (fig. 7).

Spermatids.

While the chromatin amasses at the poles of the new spermatids (fig. 11), the cytoplasm of the cells grows rapidly. The nuclear mass breaks up into a beaded appearance once more (fig. 12) and the cytoplasm begins to condense. There follows an unequal growth of nucleus and cytoplasm now, the nucleus gaining on the cytoplasm, as seen in figures 13, 14, 15. At the same time the chromatin moves to the nuclear boundary, where it forms a darkly staining band. The cytoplasm begins to elongate at one side of the nucleus until a condition seen in figure 17 is reached. The remainder of the marked metamorphosis of the cell consists of a condensing of material until a small, elongated nuclear head remains, and a long vibratile fiber. (Fig. 25.)

Melanoplus differentialis.

A glance at plate 31 will show that *differentialis* follows very closely the description for *femur-rubrum.* There are from one to two and occasionally three rings seen in an equatorial plate of the first spermatocyte, and there is a tendency for most of the larger chromosomes to form rings in the prophase. Thus we may see three or four rings at one time. *Differentialis* is the most favorable material of the *Melanoplus* group thus far investigated. The chromosomes are a little larger than *femur-rubrum* and have the advantage of not crowding so much.

Melanoplus atlanis.

Figures 1 to 12, plate 32, show spermatocyte divisions in *M. atlanis.* A comparison of this species with the two preceding shows that the chromosomes are more irregular in form, but as to general behavior, numbers, etc., are the same.

Melanoplus packardii.

Plate 32, figures 13 to 24, show a few of the stages of division in *M. packardii.* The chromosomes of this group are smaller than any of the preceding, but in general behavior and form are similar.

SUMMARY AND DISCUSSION.

In the four species of *Melanoplus* set forth in this paper we find a close similarity in form and behavior of the chromosomes in the different generations, and an identity in number throughout. That is to say, twenty-three spermatogonial chromosomes and twelve spermatocyte. The counts in the female cells did not give more than twenty-two, but there is much room for doubt, owing to crowded plates. The results of this paper tally with those on *Melanoplus bivitatus* Say ('08).

Femur-rubrum and atlanis are so similar in body form that they are easily mistaken for each other. Size, coloring and shape are alike, the distinction lying in the podical plates only. The chromosome complexes of the two species are, however, not as similar as those of *femur-rubrum* and *differentialis*. Between the two latter is a marked difference in body size, yet the chromosomes are the same in size and shape. There is no size difference in the chromosomes of *M. femur-rubrum* and *M. atlanis*, but the form of *atlanis* chromosomes is more irregular.

On the whole, the close resemblance between the complexes of the five species examined supports the belief that there is correlation between chromosomes and body characters.

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EXPLANATION OF PLATES.

All drawings made with a 12 ocular, one-twelfth oil immersion obj. and Abbe camera lucida; reproduced at a magnification of about 2000 diameters. Drawings are arranged in the order of their development as nearly as possible, beginning with spermatocytes and ending with spermatogonia. Figures 1-65, plates 28, 29, 30, represent *Melanoplus femur-rubrum;* plate 31, *Melanoplus differentialis;* and plate 32, figures 1-12, *Melanoplus atlanis,* figures 13-24, *Melanoplus packardii.*

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EXPLANATION OF PLATES.

PLATE 1.

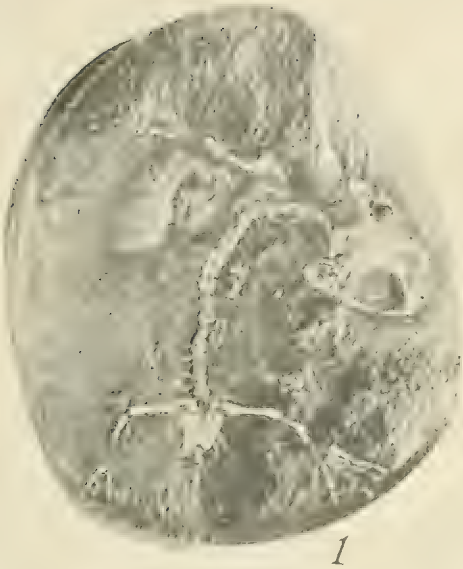
FIG. 1.—The nodule containing the remains of *Amphibamus grandiceps* Cope. X 1. (No. 1234, Yale Museum.)

FIG. 2.—The opposite half of the same nodule. X 1.

FIG. 3.—*Erierpeton branchialis* Moodie. X 1. (No. 801, Yale Museum.)

FIG. 4.—*Cephalerpeton ventriarmatum* Moodie. X 1. (No. 796, Yale Museum.)

PLATE 1.



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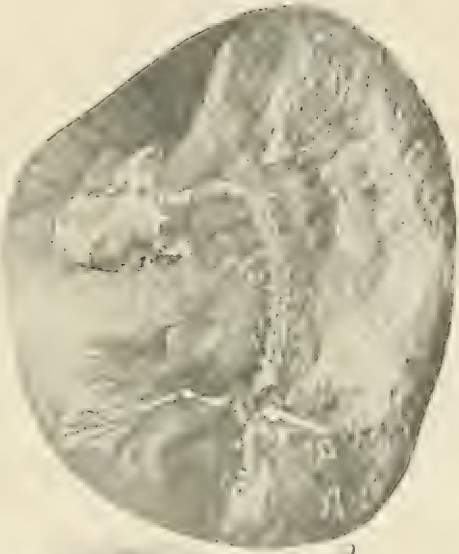

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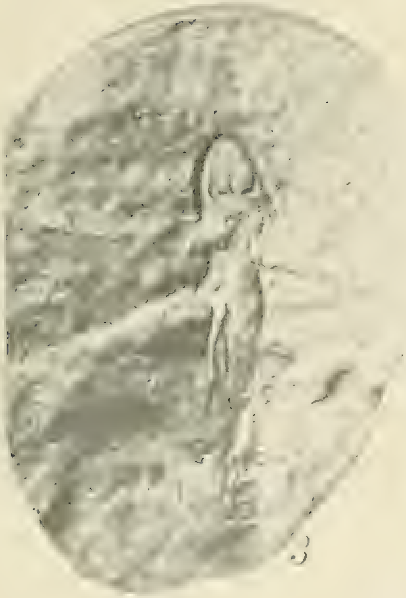

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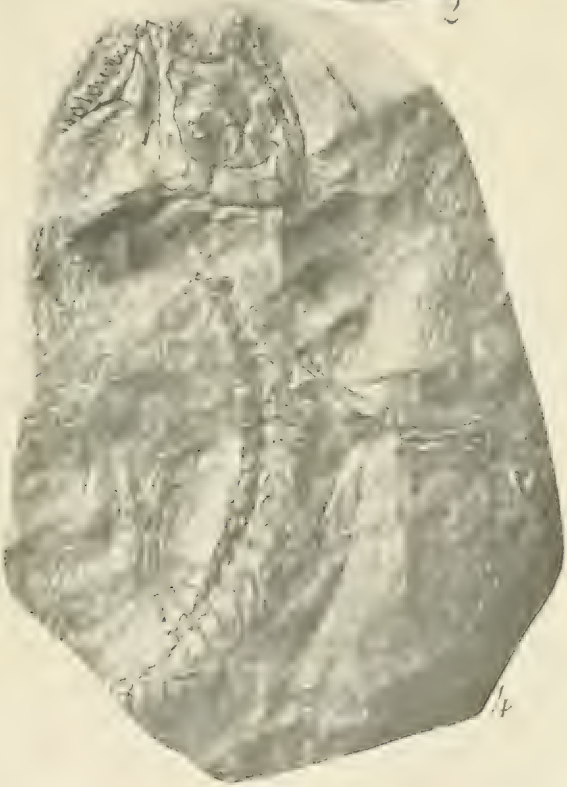

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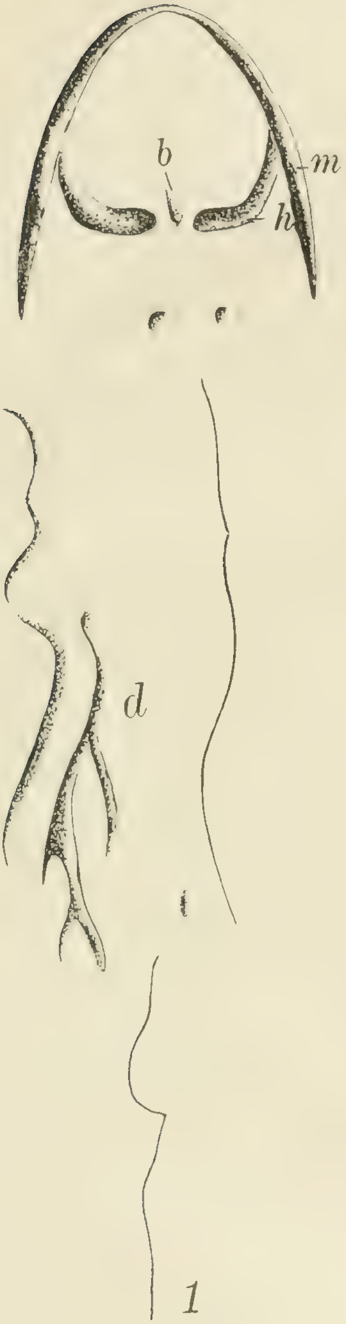
PLATE 2.

FIG. 1.—Drawing of the specimen of *Erierpeton branchialis* Moodie. X 3. *b*, basibranchial; *d*, body impressions; *h*, hypohyal; *m*, mandible.

FIG. 2.—The arm skeleton of *Erpetobrachium mazonensis* Moodie. X 2. *cl*, clavicle; *h*, humerus; *r*, radius; *s*, scapula; *u*, ulna.

FIG. 3.—Rib of *Mazonerpeton costatum* Moodie. X 4. (Yale Museum.)

PLATE 2.



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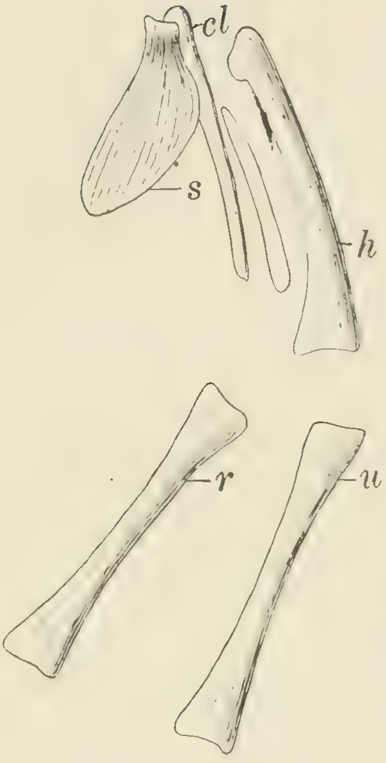

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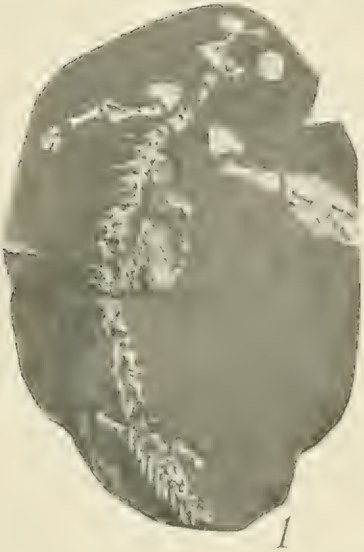
PLATE 3.

FIGS. 1 AND 2.—The halves of the nodule containing the skeleton of *Mazonerpeton longicaudatum* Moodie. X 1. (No. 795, Yale Museum.)

FIG. 3.—The larger specimen of *Eumicrerpeton parvum* Moodie. X 1. (No. 803, Yale Museum.) Shows impressions of intestines.

FIG. 4.—The smaller specimen of *Eumicrerpeton parvum* Moodie. X 1. (No. 802, Yale Museum.) Shows mold of intestines.

PLATE 3.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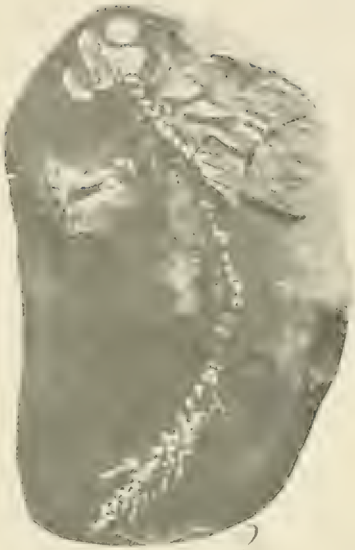
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PLATE 4.

An ideal restoration of *Eumicrerpeton parvum* Moodie in surroundings of Carboniferous plants as they are preserved in the nodules from Mazon Creek, where they are associated with the Amphibia. The size of the animal in life was from an inch and a half to two inches in length.

PLATE 4.

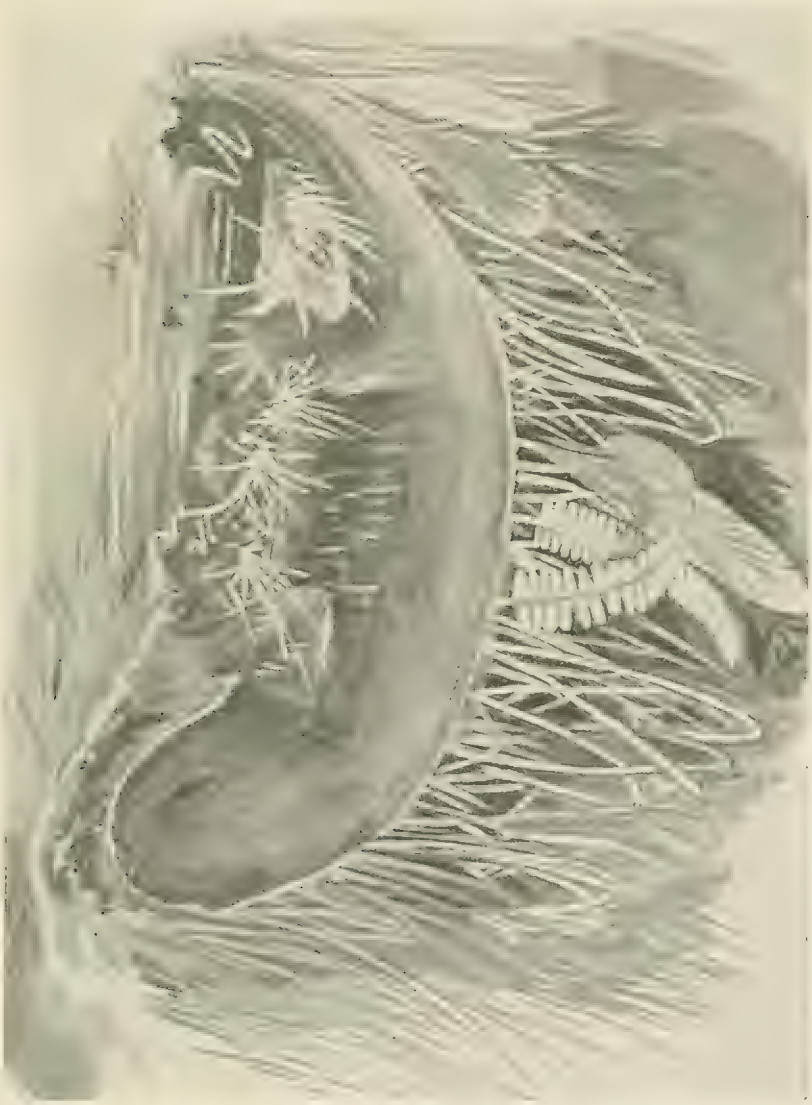


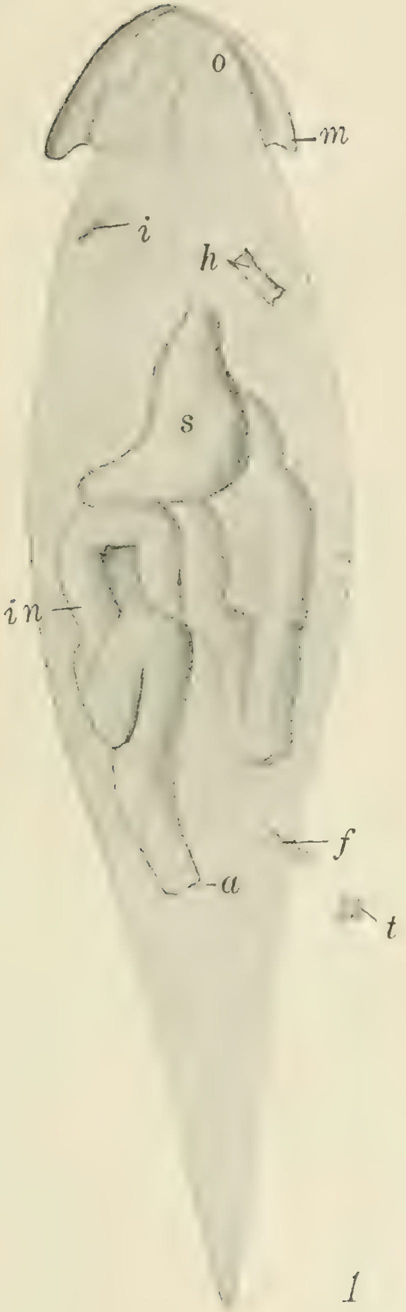
PLATE 5.

FIG. 1.—Drawing of the third specimen of *Eumicrerpeton parvum* Moodie. X 4. (No. 4400, United States National Museum.) *a*, anus; *f*, femur; *h*, humerus; *i*, interclavicle (?) clavicle; *in*, intestine; *m*, mandible; *o*, orbit; *s*, stomach; *t*, tibia and fibula.

FIG. 2.—Drawing of the skeleton of *Amphibamus thoracatus* Moodie. X 2. (No. 4306, United States National Museum.) *c*, clavicle; *h*, humerus; *i*, interclavicle; *o*, orbit; *r*, radius; *v*, vertebra.

FIG. 3.—Cope's drawing of the type specimen of *Amphibamus grandiceps* Cope. X 1 ca. (Geol. Survey Illinois, vol. 2, pl. 32.)

PLATE 5.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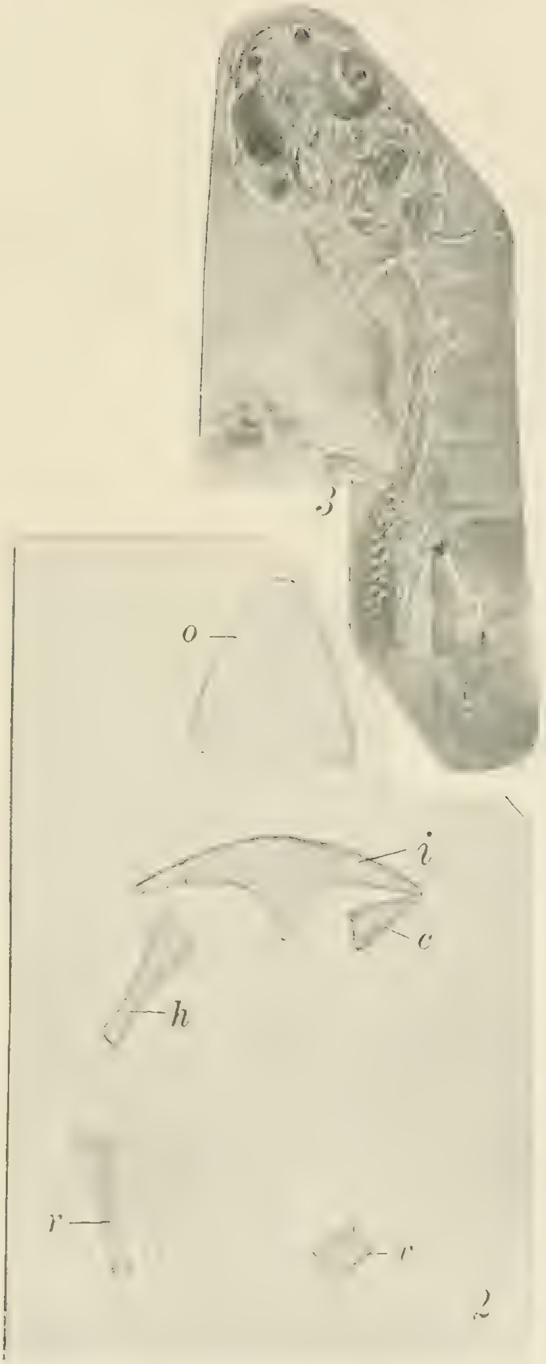
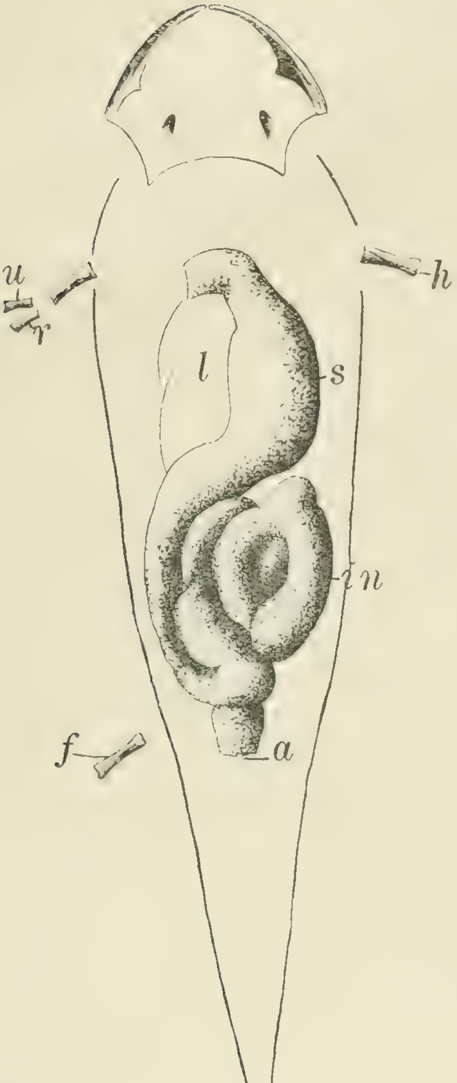


PLATE 6.

FIGS. 1 AND 2.—Drawings of the Yale specimens of *Eumicrerpeton parvum* Moodie. (X 4½ and 4.) *a*, anus; *d*, dorsal lateral line; *f*, femur; *h*, humerus; *in*, intestine; *l*, liver impression; *m*, median lateral line of the tail; *o*, orbit; *p*, parietal; *r*, radius; *s*, stomach; *u*, ulna.

PLATE 6.



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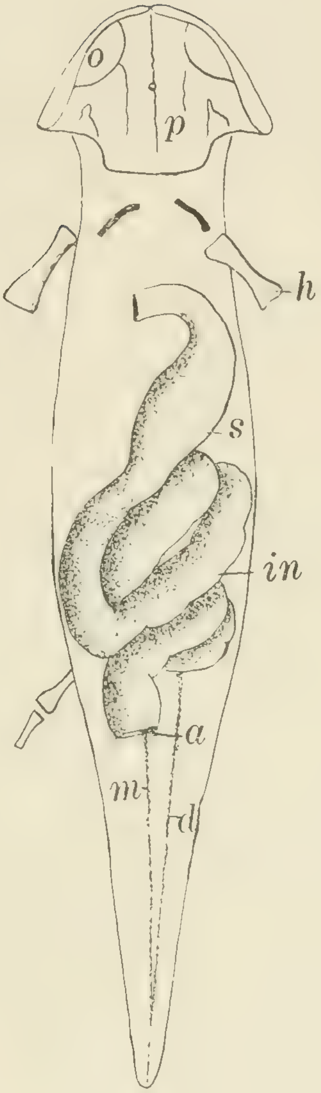

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PLATE 7.

FIG. 1.—Drawing of the skeleton of *Amphibamus grandiceps* Cope, in Yale Museum. (X 1¼.) *C*, carpus; *cl*, clavicle; *cr*, caudal ribs; *cv*, caudal vertebræ; *f*, femur; *h*, humerus; *k*, a piece of skin; *il*, ilium; *o*, orbit; *r*, radius; *ri*, ribs; *s*, scapula; *sc*, sclerotic plates of the eye; *u*, ulna; *vs*, ventral scutellæ; *t*, tibia; *T*, tarsus.

FIG. 2.—Drawing of the skeletal remains of *Cephalerpeton ventriarmatum* Moodie. X ½. Lines to the right of the skeleton represent boundary of the nodule. *a*, prefrontal; *cl*, clavicle; *d*, dentary; *h*, humerus; *f*, frontal; *j*, jugal; *m*, maxilla; *o*, orbit; *p*, phalanges; *pa*, parietal; *pr*, postorbital; *r*, radius; *s*, sclerotic plates; *u*, ulna; *vs*. ventral scutellæ.

FIG. 3.—Drawing of the skeleton of *Mazonerpeton longicaudatum* Moodie. X 2. *C*, carpus; *cl*, clavicle; *cv*, caudal vertebræ; *f*, femur; *h*, humerus; *r*, radius; *ri*, rib; *s*, scapula; *sc*, sclerotic plates; *vs*, ventral scutellæ; *u*, ulna.

PLATE 7.

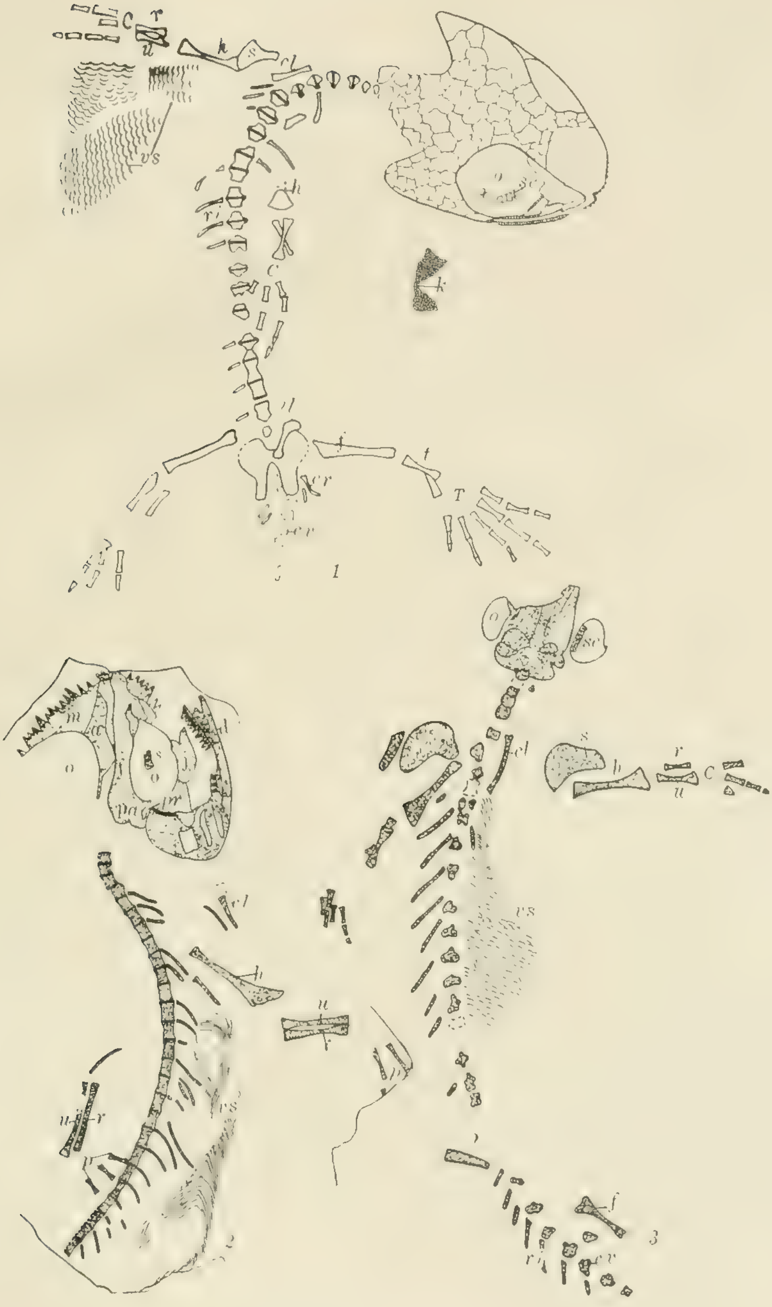
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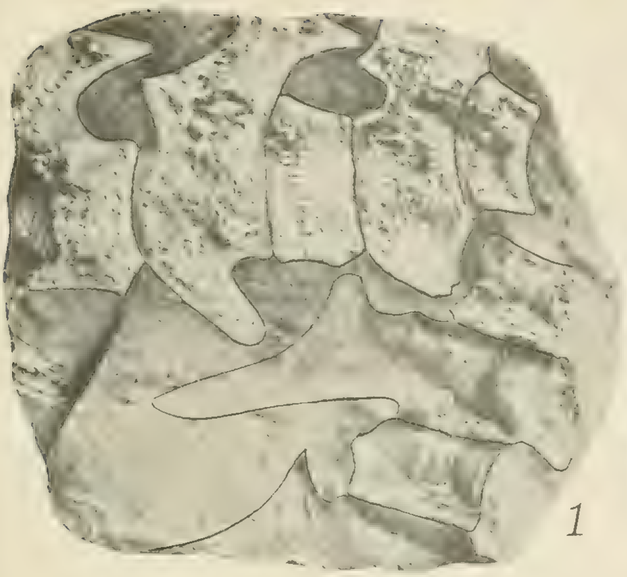
PLATE 8.

FIGS. 1 AND 2.—The two halves of the nodule containing the vertebræ of *Spondylerpeton spinatum* Moodie. X 1. (No. 793, Yale Museum.)

FIG. 3.—The arm elements of *Erpetobrachium mazonensis* Moodie. X 1. (No. 799, Yale Museum.) *h*, humerus; *r*, radius; *s*, scapula; *u*, ulna.

FIG. 4.—The skeleton of *Mazonerpeton costatum* Moodie. X 1. (No. 800, Yale Museum.)

PLATE 8.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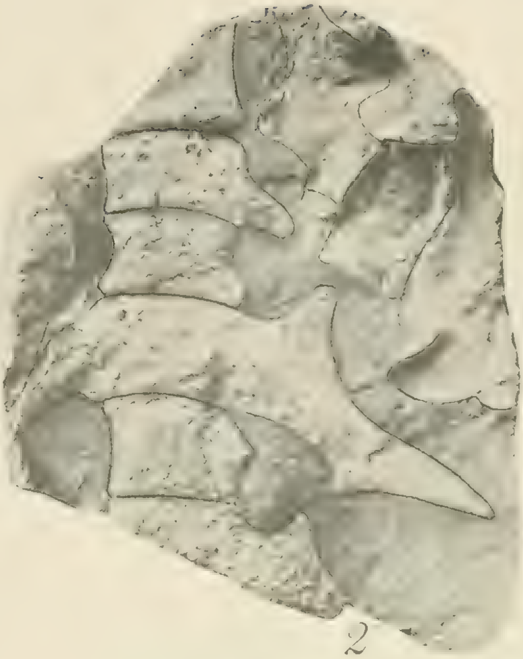
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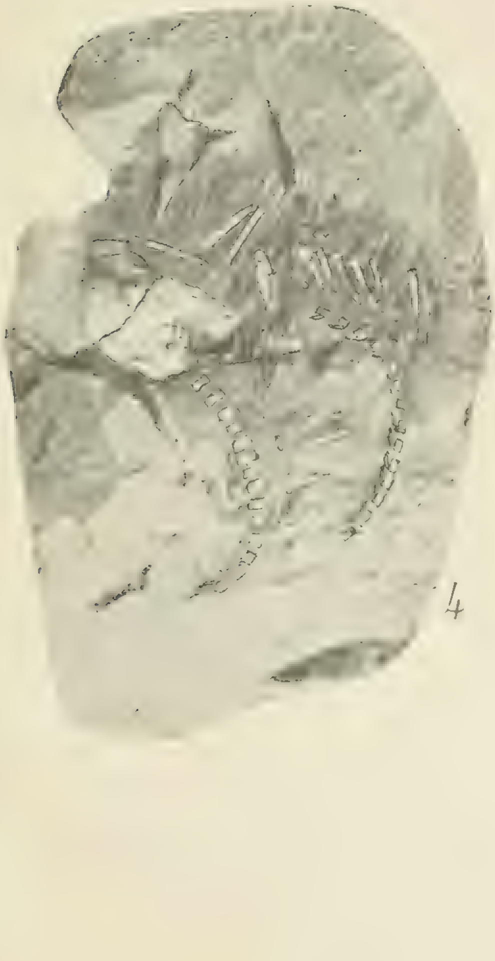
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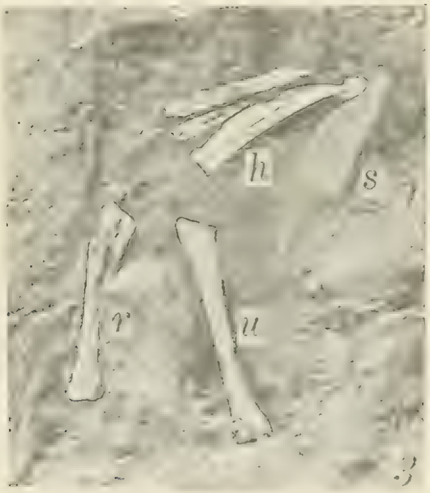

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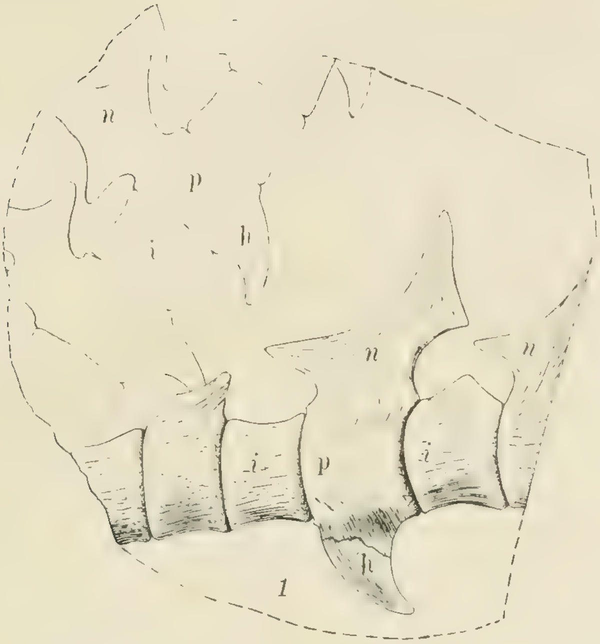
PLATE 9.

FIG. 1.—Drawing of the vertebral remains of *Spondylerpeton spinatum* Moodie. X 1. *h*, chevron; *i*, intercentrum; *n*, neural spine (neurocentrum) ; *p*, pleurocentrum.

FIG. 2.—The skeletal remains of *Mazonerpeton costatum* Moodie. X 2. *ac*, anterior caudal; *ch*, chevron; *cl*, clavicle; *cv*, caudal vertebræ; *f*, femur; *h*, humerus; *m*, mandible; *r*, ribs; *v*, vertebra; *sk*, skull.

PLATE 9.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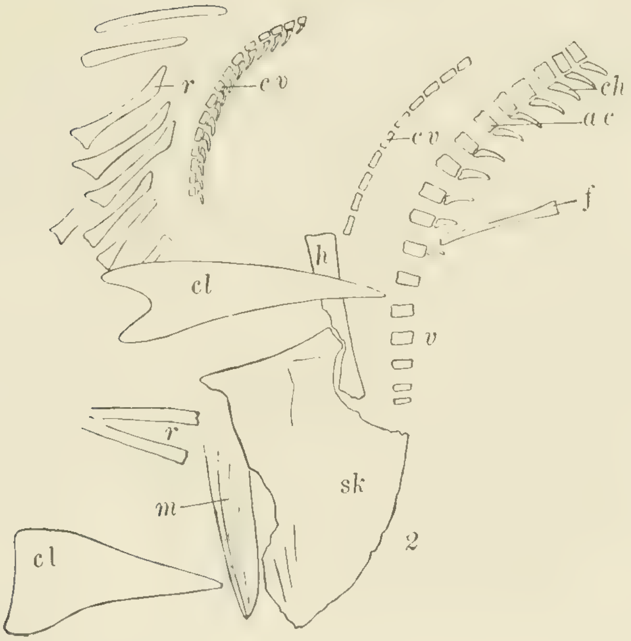
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PLATE 10.

Ideal restoration of *Mazonerpeton* crawling on calamite stems and about to feed on an *Acanthotelson*. Drawn from the fossils.

PLATE 10.

PLATE 11.

Photographs of the two halves of the nodule containing the remains of *Amphibamus grandiceps* Cope, in the possession of Mr. L. E. Daniels, of Rolling Prairie, Ind. This specimen shows the form of the body and a large part of the skeletal structure. The orbits are blackened with the *pigmentum nigrum* of the iris.

PLATE 11.

PLATE 12.

Restoration of the skeleton of *Amphibamus grandiceps* Cope, on the basis of the Yale specimen, the specimen in Mr. Daniels' possession, and Cope's drawing of the type. The outline of the body is restored from the specimen in the possession of Mr. Daniels, as may be seen by referring to plate 11. *Ca*, capus; *Cl*, clavicle; *E*, epiotic plate; *Fe*, femur; *Fi*, fibula; *F*, frontal; *Hu*, humerus; *Ic*, interclavicle; *Il*, ilium; *Is*, ischium; *J*, jugal; *L*, lachrymal; *M*, maxilla; *N*, nasal; *O*, orbit; *P*, parietal; *Pf*, postfrontal; *Po*, postorbital; *Pr*, prefrontal; *Px*, premaxilla; *Pu*, pubis; *Qj*, quadratojugal; *R*, radius; *So*, supraoccipital plate; *Sq*, spuamosal; *sr*, sacral rib; *St*, supratemporal (squamosal) (paraquadrate); *Sc*, scapula; *Ta*, tarsus; *Ti*, tibia.

PLATE 12.

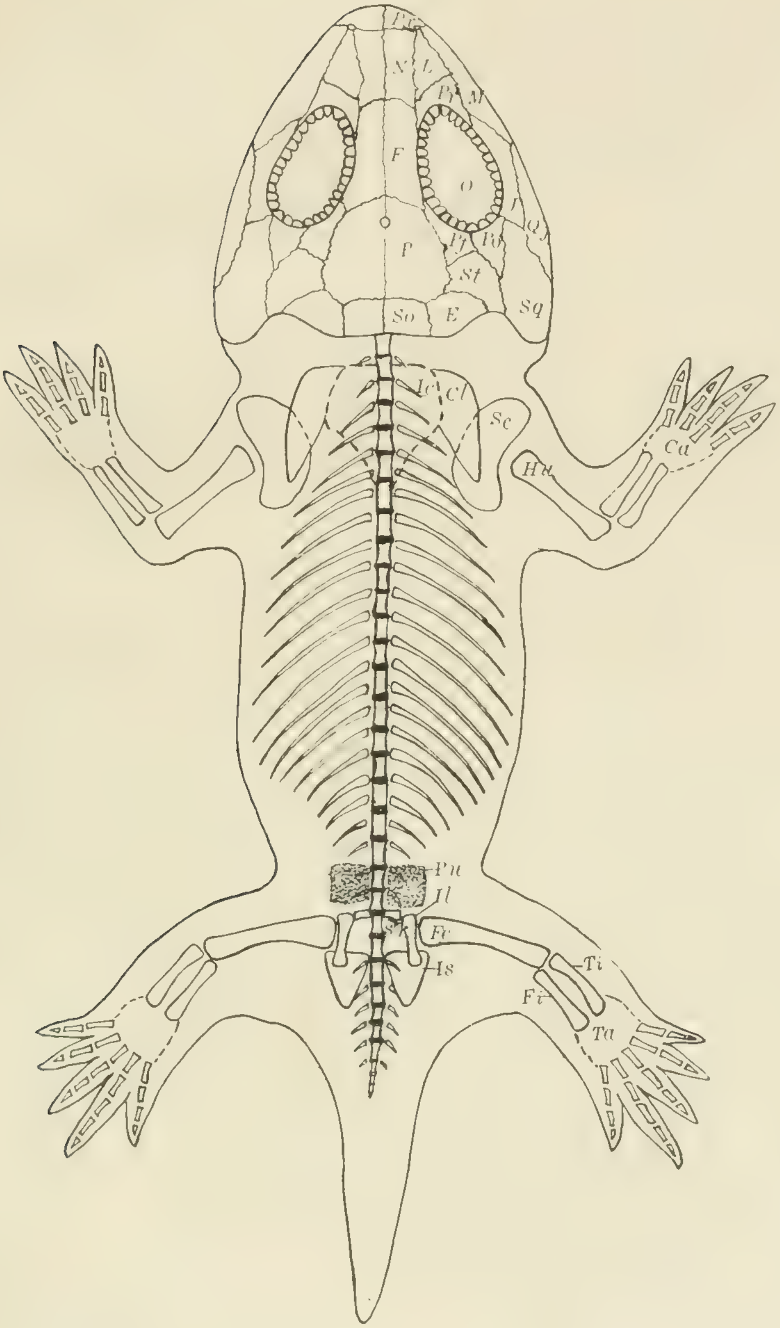
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PLATE 13.

Restoration of *Amphibamus grandiceps* Cope as it probably appeared in life.

PLATE 13.

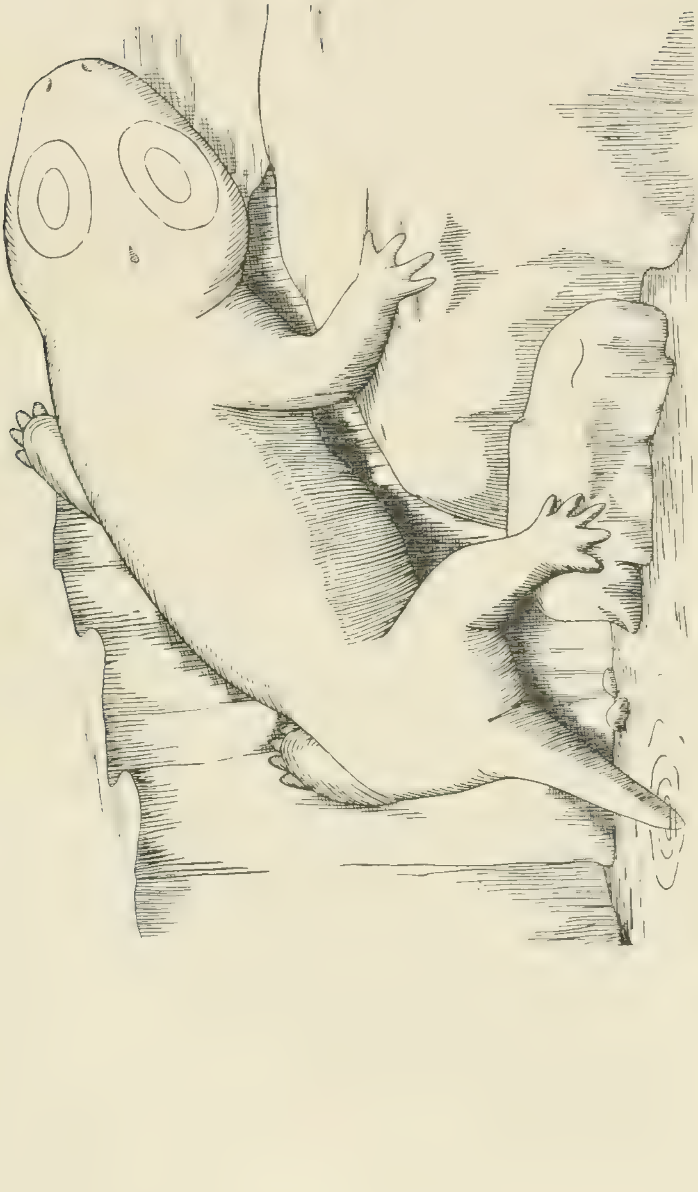


PLATE 14.

Branchiosauria from the Upper Carboniferous (Commentry beds) of France.

Figs. 1, 2, 3, 8 and 9.—*Branchiosaurus (Protriton) fayoli* Thevenin. Specimens showing the external form of the body and portions of the skeleton.

Fig. 4.—An enlarged view of the ventral scutellæ of *Branchiosaurus fayoli* Thevenin. X 15.

Fig. 5.—The entire form of the ventral scutellæ of the same species.

Fig. 6.—A specimen of *Branchiosaurus fayoli* Thevenin, showing sclerotic plates and external branchiæ. X 2.

Fig. 7.—A complete specimen of *Branchiosaurus (Protriton) petrolei* Gaudry, showing external branchiæ. From the Lower Permian of Autun. X 1. All figures after Thevenin.

PLATE 14.

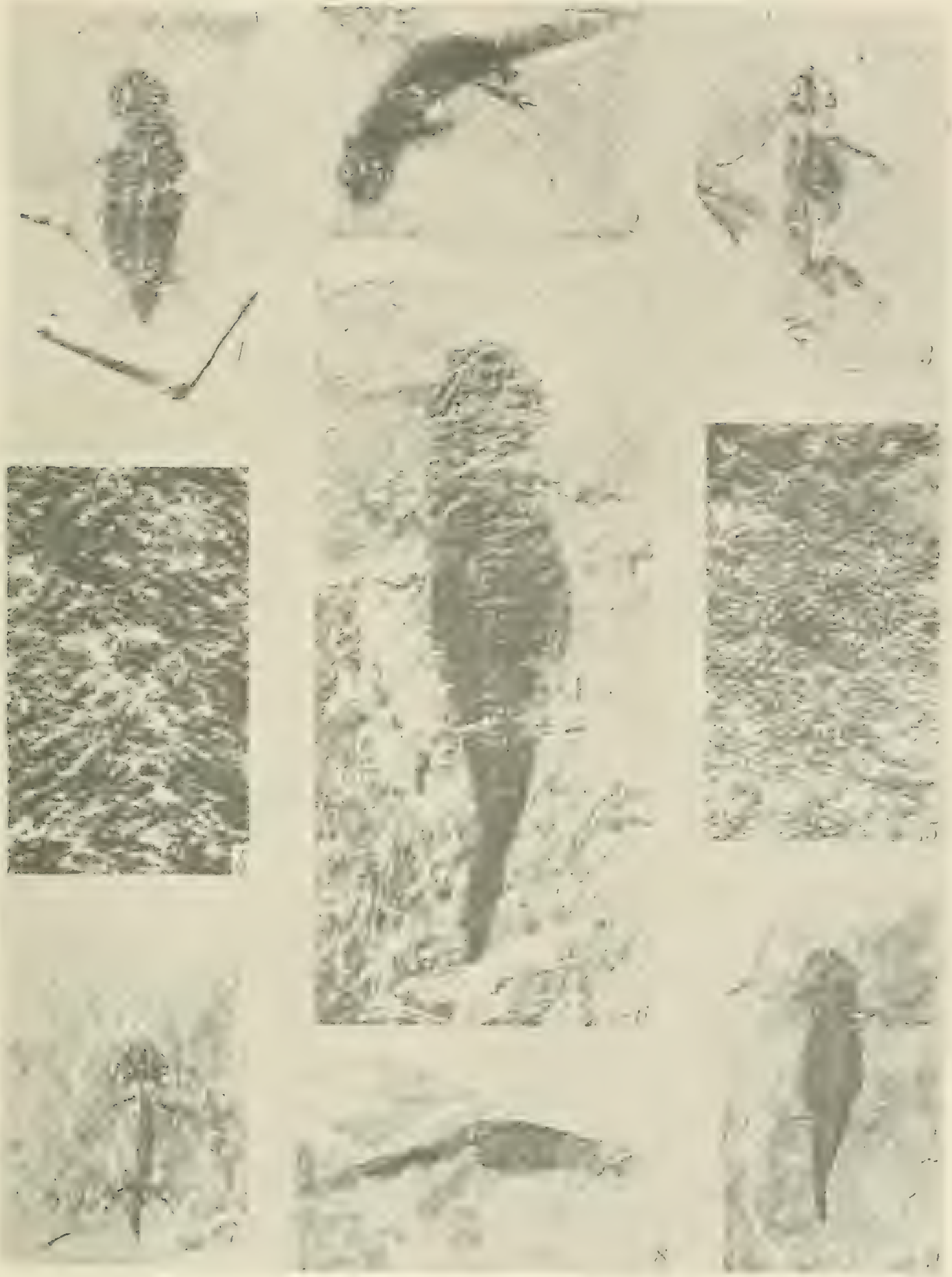


PLATE I.

FIG. 1.—Flowering branch of *Krameria canescens*. × ⅗.

FIG. 2.—Root showing knotty crown and numerous secondary roots. × ½.

PLATE I. -15

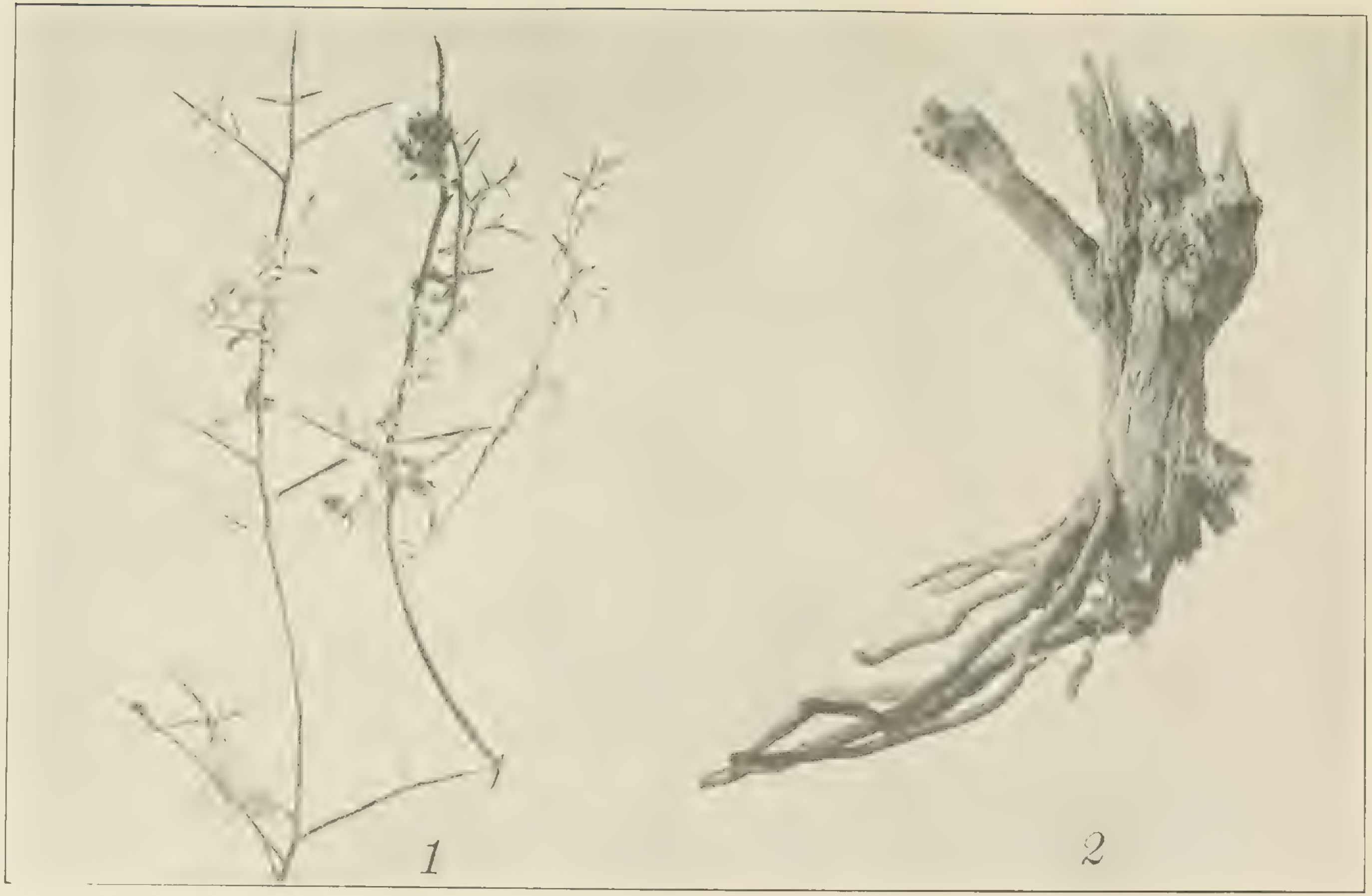


PLATE II.

FIG. 3.—Cross section of stem, showing one year's growth: *e*, epidermis, with the cuticle represented by the outer black circle; *p*, palisade; *y*, thin-walled parenchyma of primary cortex; *n*, pericycle; *d*, endodermis, *a*, bast fibers; *o*, phloëm; *x*, xylem; *r*, medullary ray; *m*, medulla. × 78.

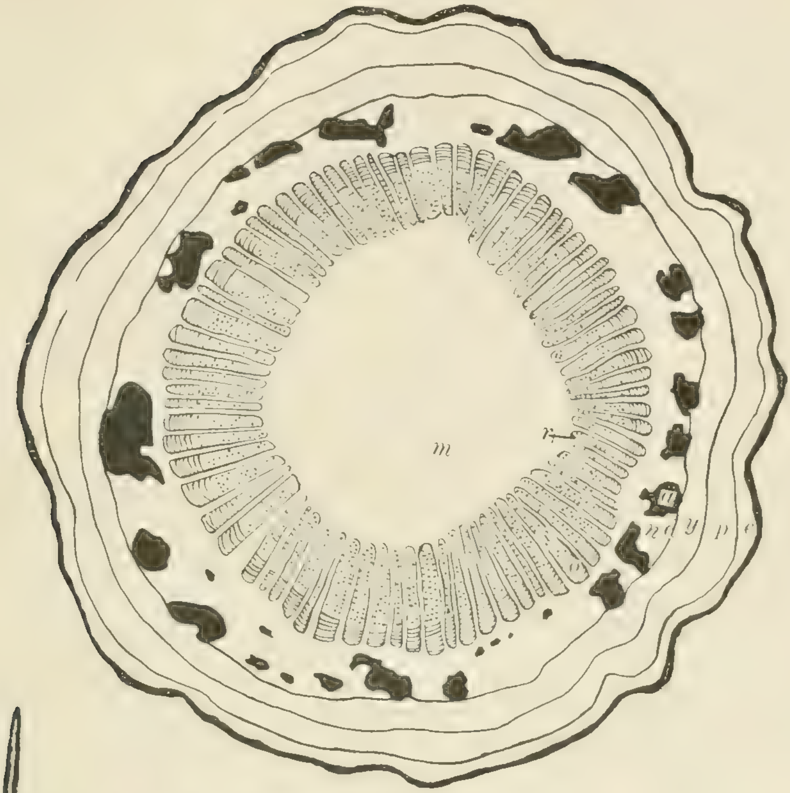
FIG. 4.—Cross section of secondary root: *s*, brown scaly cork; *g*, cork, the cells filled with reddish-brown coloring matter; *v*, phellogen; *l*, thin-walled parcenchyma; *f*, phloëm; *w*, xylem; *u*, cambium. × 10.

FIG. 5.—Small portion of stem: *c*, cambium; *t*, tracheids; *r*, medullary ray. × 475.

FIG. 6.—Bast fibers, and parenchyma of the pericycle of stem. × 325.

FIG. 7.—Tracheid from stem. × 325.

PLATE II.



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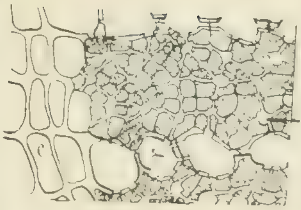

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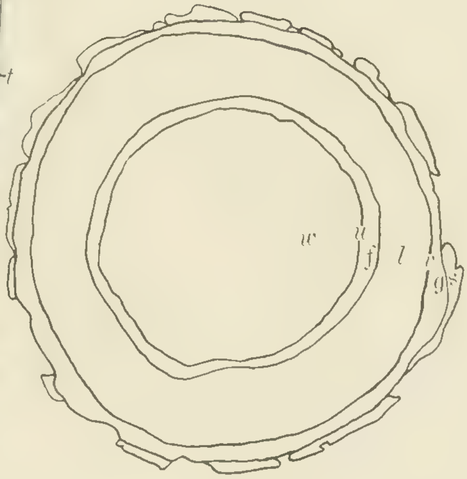

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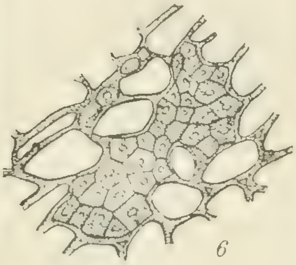

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PLATE III.

FIG. 8.—Cross section of a small portion of a stem 4.6 mm. in diameter: *e*, epidermis, the cuticle represented by the black outer band; *x*, palisade; *q*, thin-walled parenchyma of primary cortex; *y*, pericycle; *c*, cambium; *w*, water tubes; *m*, medullary rays; bast fibers are shown in the phloëm and pericycle by black dots. × 65.

FIG. 9.—Longitudinal radial section of stem: *u*, spiral water tube; *l*, reticulate water tube; *f*, medulla. × 288.

FIG. 10.—Portion of phloëm: *b*, bast fibers; *r*, medullary rays. × 325.

FIG. 11.—Longitudinal radial section of stem phloëm: *b*, bast fiber; *n*, undivided mother cells. × 325.

FIG. 12.—Stem tracheids. × 288.

FIG. 13.—Stem bast fiber. × 90.

FIG. 14.—Tangential section of stem cork. × 325.

PLATE III.

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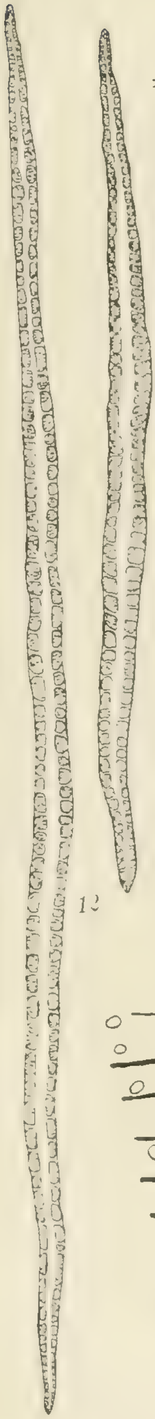
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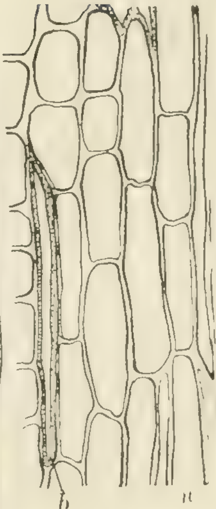

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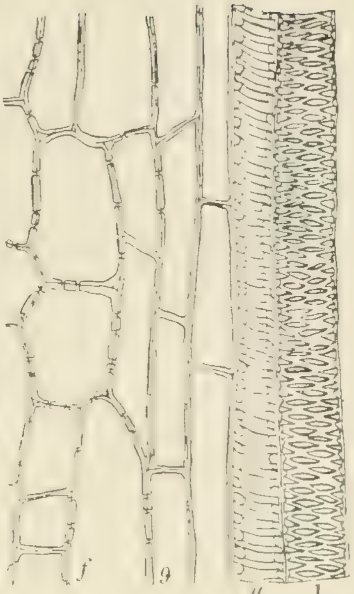

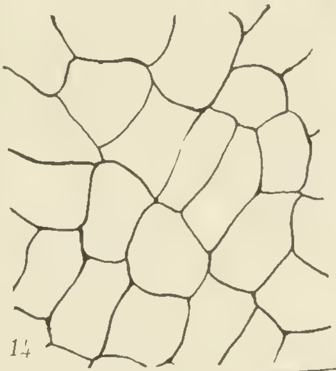
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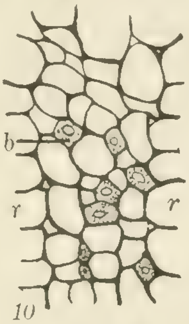

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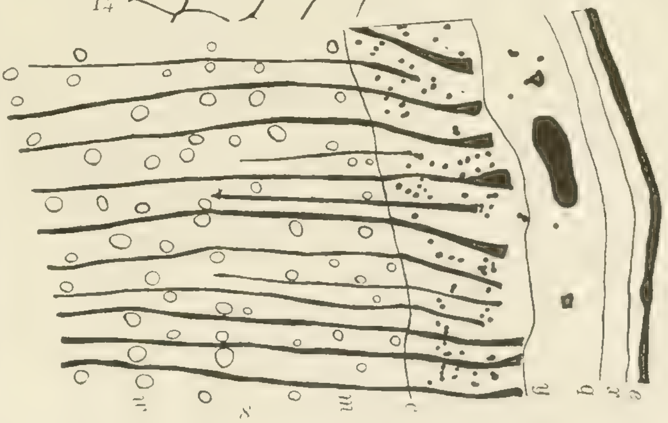


PLATE IV.

FIG. 15.—Portion of stem: *t*, trichomes; *e*, epidermis; *y*, palisade; *bf*, bast fibers. × 325.

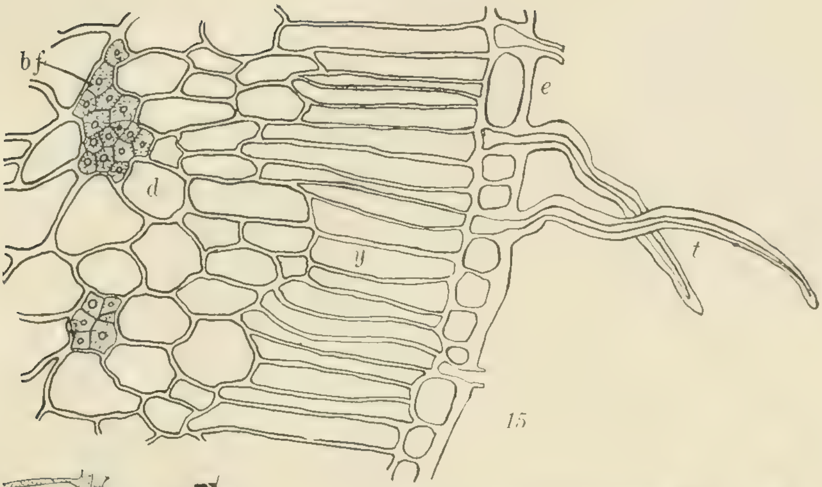
FIG. 16.—Surface view of stem epidermis: *s*, stoma; *h*, scar of trichome. × 175.

FIG. 17.—Cross section of stem, one year old, showing longitudinal section of a stoma: *st*, *p*, palisade; *cl*, chloroplasts; *g*, guard cell; *i*, air chamber; the cuticle is shaded black. × 475.

FIG. 18.—Longitudinal section of stem, showing cross section of a stoma. × 475.

FIG. 19.—Longitudinal radial section of stem cambium: *c*, cambiform cell of the phloëm. × 325.

PLATE IV.



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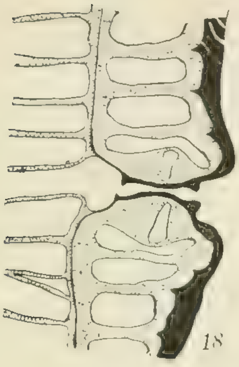

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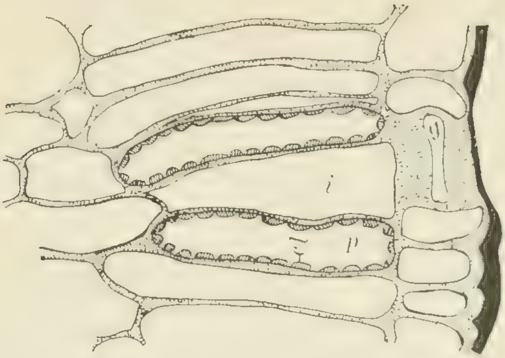

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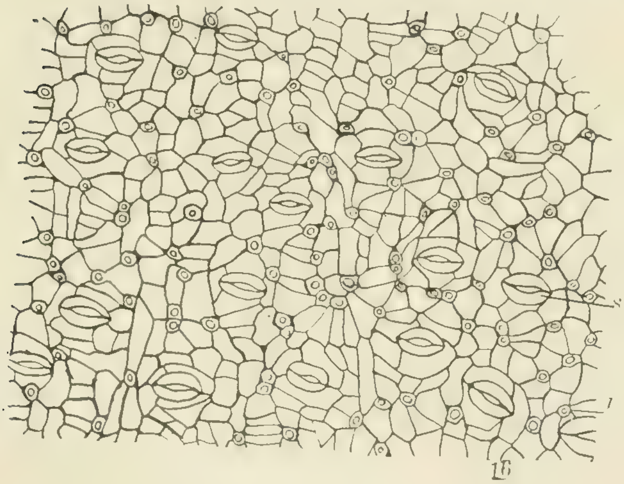

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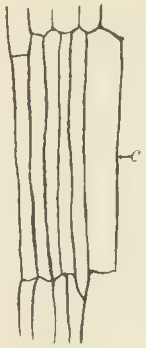

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PLATE V.

FIG. 20.—Longitudinal radial section of stem medullary ray. × 475.

FIG. 21.—Longitudinal tangential section of stem medullary ray. × 475.

FIG. 22.—Longitudinal tangential section of stem wood parenchyma. × 475.

FIG. 23.—Cross section of stem tracheids; and wood parenchyma, *w*. × 475.

FIG. 24.—Cross section of a small portion of a root; *p*, phloëm; *c*, cambium; *m*, medullary ray; *x*, xylem. × 325.

FIG. 25.—Pith from cross section of one-year-old stem. × 325.

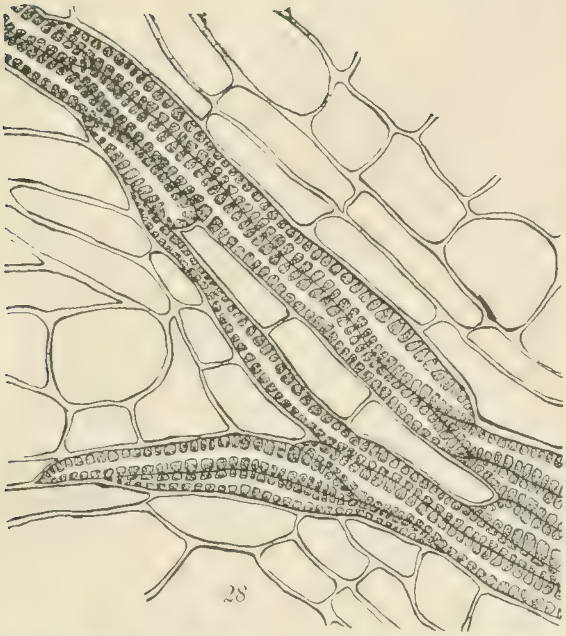
FIG. 26.—Longitudinal section of a stem water tube. × 288.

FIG. 27.—Surface view of a small portion of a water tube. × 288.

FIG. 28.—Tracheids and parenchyma cells from a leaf cut longitudinally through the broad diameter. × 475.

FIG. 29.—Cambiform cells from the phloëm of a leaf, cut longitudinally through the narrow diameter. × 325.

PLATE V.



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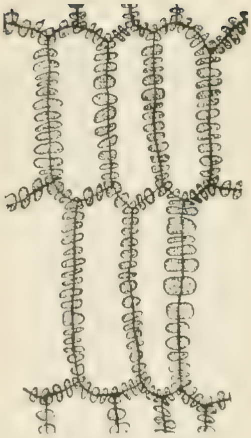

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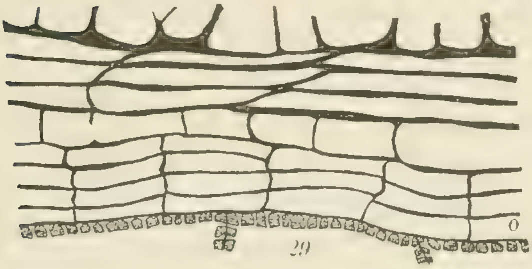

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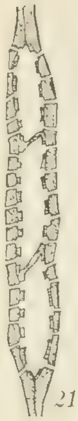

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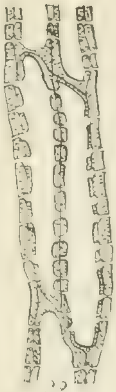

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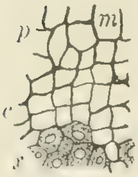

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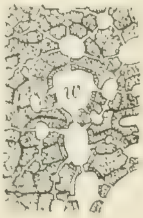

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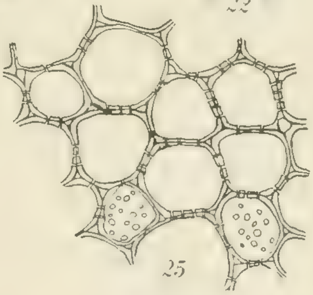

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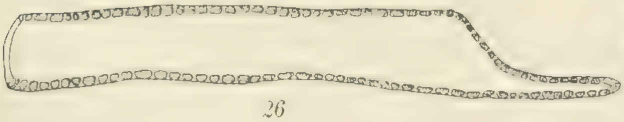

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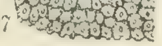

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PLATE VI. : 2c

FIG. 30.—Cross section of a secondary root; *d*, brown scaly cork; *c*, cork, the cells colored reddish brown; *k*, phellogen; *g*, cambium; the bast fibers are shown by black dots and the water tubes by circles. × 41.

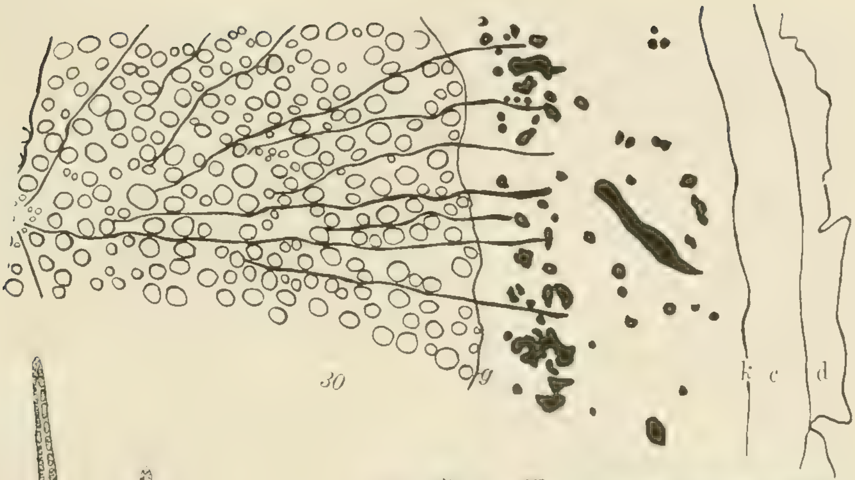
FIG. 31.—Cross section of xylem of root; *wt*, water tube; *t*, tracheid; *mr*, medullary ray. × 375.

FIGS. 32, 33, 34.—Tracheids from root. × 288.

FIG. 35.—Cross section of a portion of a root; *o*, cork; *n*, phellogen; *y*, thin-walled parenchyma. × 325.

FIG. 36.—Bast fiber from root. × 180.

PLATE VI.



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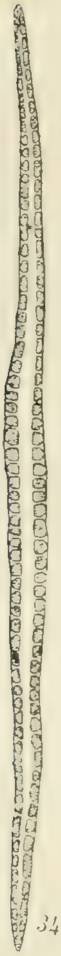

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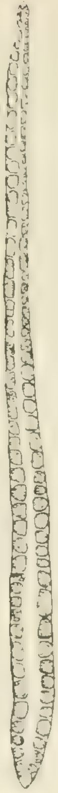

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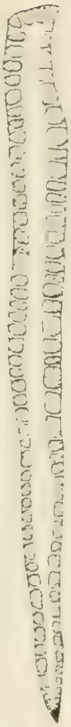

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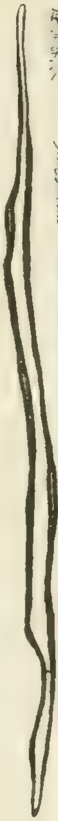

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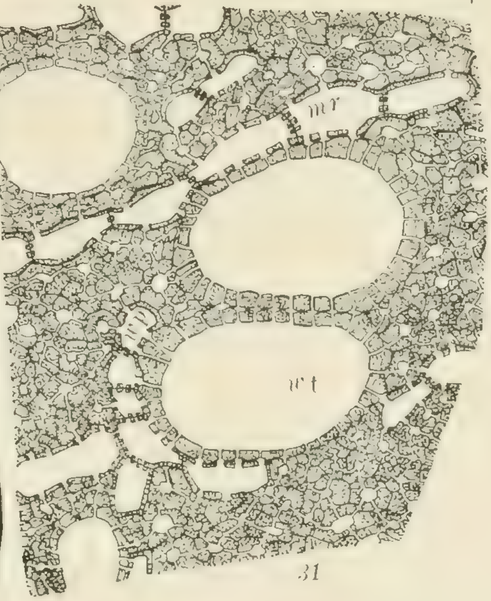

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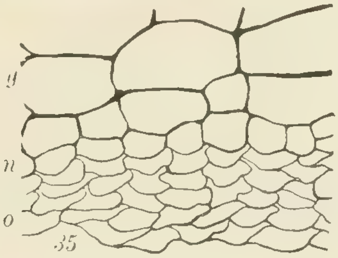

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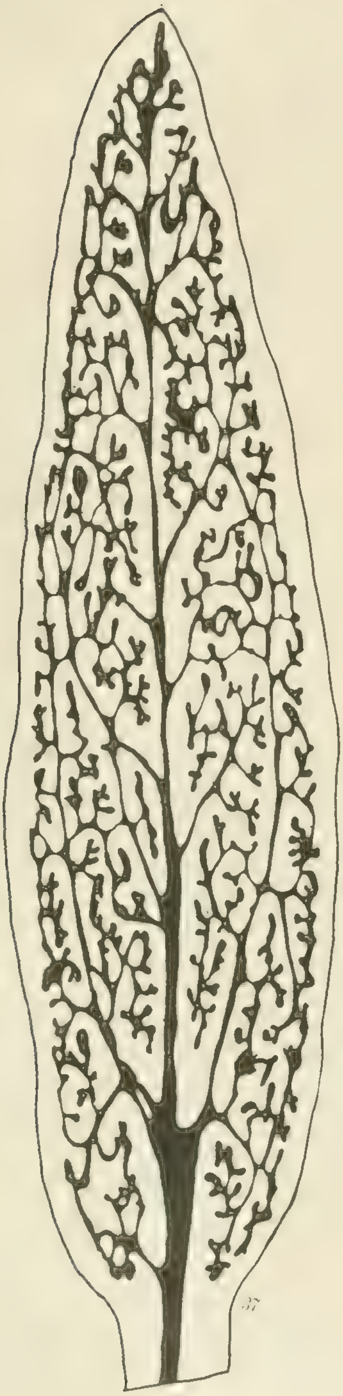
PLATE VII.

FIG. 37.—A bleached leaf, showing the venation. × 22.

FIG. 38.—Cross section of a leaf. × 120.

PLATE VII.



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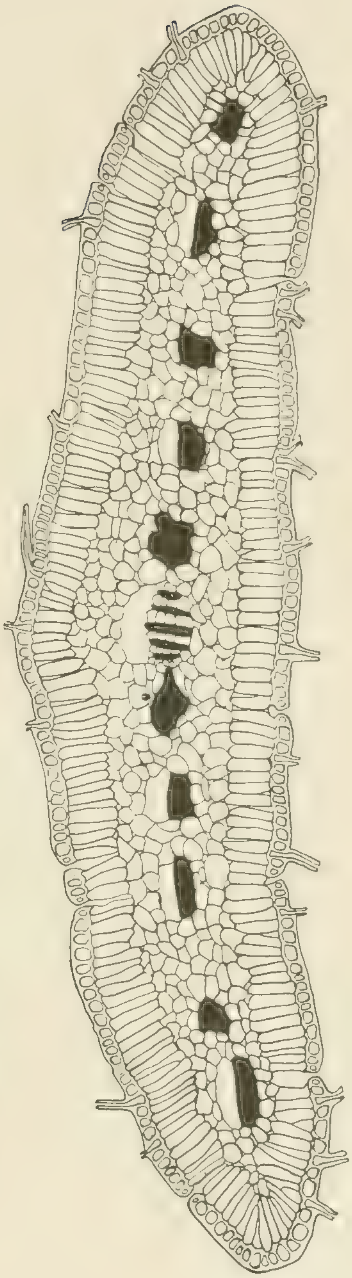

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PLATE VIII. - 22

Fig. 39.—Surface view of leaf epidermis, upper surface: *s*, stoma; *h*, scar of trichome. × 175.

Fig. 40.—Surface view of leaf epidermis, under surface: *s*, stoma; *h*, scar of trichome. × 175.

Fig. 41.—Surface view of leaf palisade. × 325.

Fig. 42.—Cross section of a stoma, from a leaf cut longitudinally through the narrow diameter. × 475.

Fig. 43.—Cross section of the midvein of a leaf: *ph*, phloëm; tracheids shown by stippling. × 325.

Fig. 44.—Storage tracheids, from the ultimate ends of the veins of a leaf cut longitudinally through the narrow diameter. × 275.

Fig. 45.—Storage tracheids from the ultimate end of a vein of a leaf cut longitudinally through the broad diameter. × 475.

Fig. 46.—Spiral tracheid from a leaf. × 475.

Fig. 47.—Trichomes from a leaf. × 175.

Fig. 48.—Starch grains from the thin-walled parenchyma of a root. × 325.

Fig. 49.—Starch grains from the pith, medullary rays, pericycle, and primary cortex of a stem. × 325.

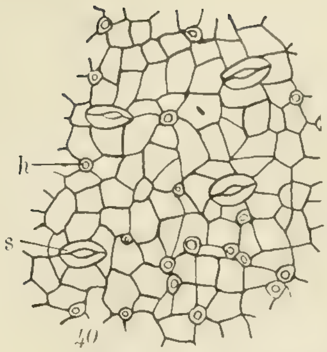
Fig. 50.—Calcium oxalate crystals from the bark of a root. × 325.

Fig. 51.—Calcium oxalate crystals: *a*, from the cortex of stem; *b*, from a leaf. × 325.

Fig. 52.—Calcium oxalate crystals from the pith of a stem. × 325.

Fig. 53.—Crystals of calcium oxalate from the phloëm of a stem. × 325.

PLATE VIII.



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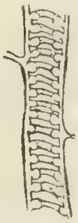
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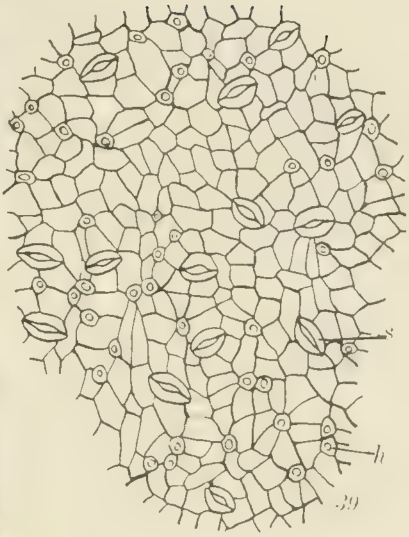

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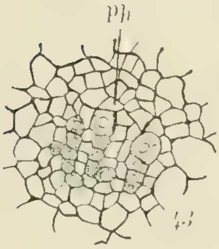

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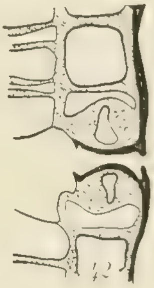
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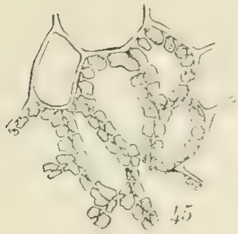
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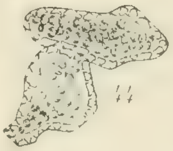
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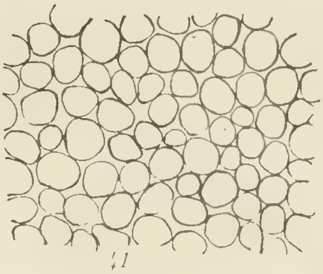
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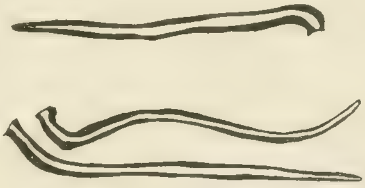
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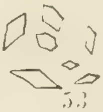
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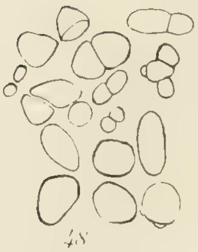
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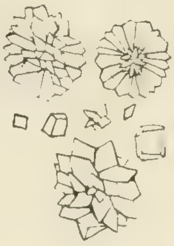
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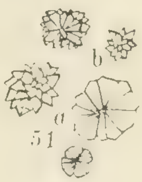

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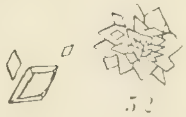
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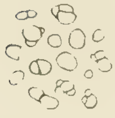
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EXPLANATION OF PLATE.

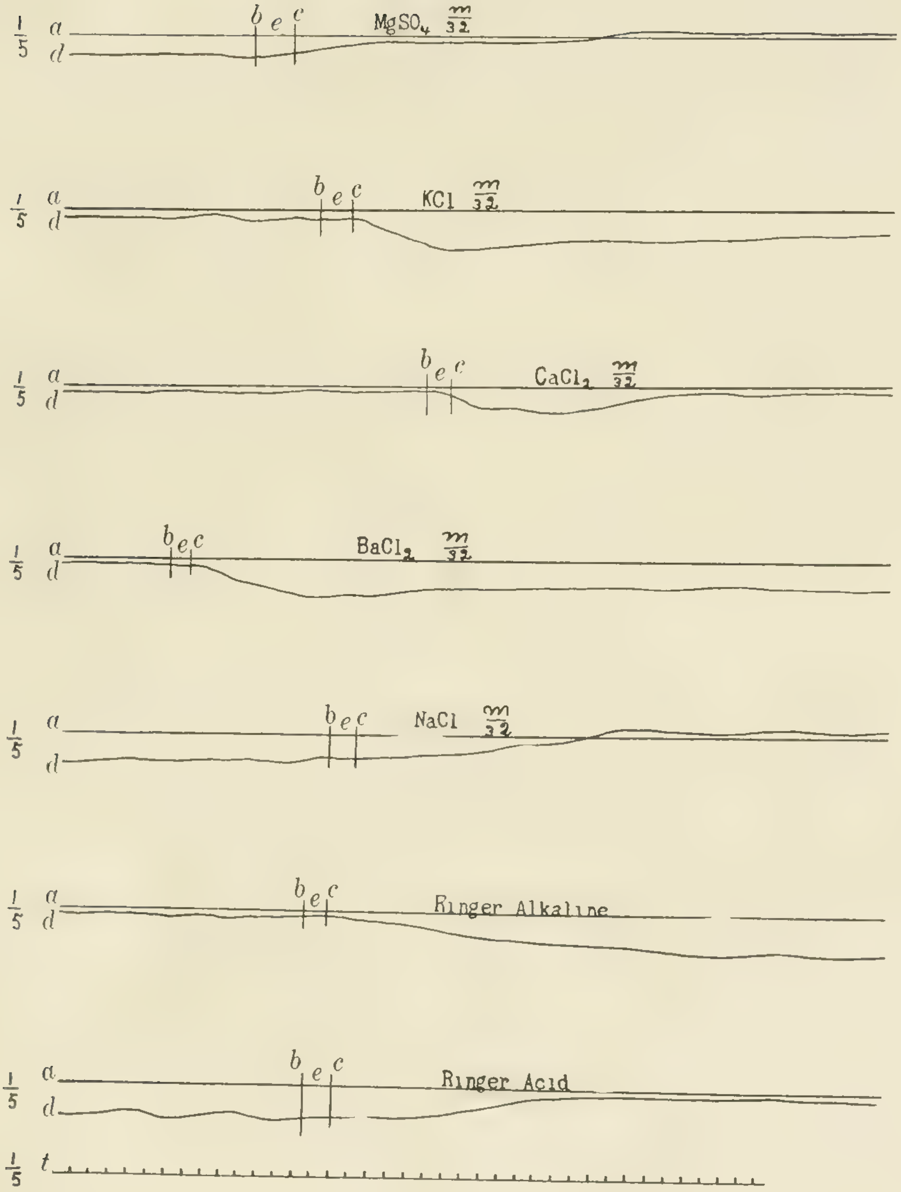
PLATE 23.

Characteristic curves of m/32 solutions. Curve approaches abscissa in relaxation.

a, Abscissa line.
d, Effect of neutral Ringer.
b, Neutral Ringer withdrawn.
e, Receptacle empty.
c, Salt solution added.
t, Time curve in 1/2 minutes.

Time and curves are reduced 1/5.

PLATE 23.



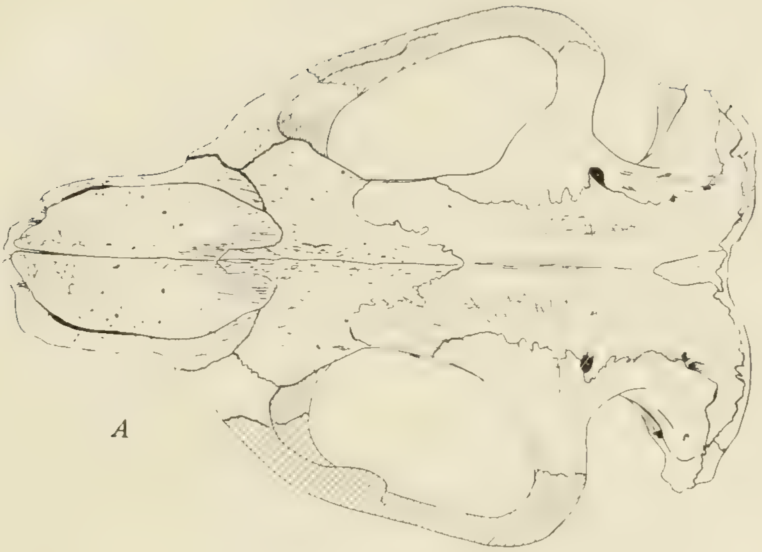
1/5 a d
b e c
MgSO$_4$ m/32
1/5 a d
b e c
KCl m/32
1/5 a d
b e c
CaCl$_2$ m/32
1/5 a d
b e c
BaCl$_2$ m/32
1/5 a d
b e c
NaCl m/32
1/5 a d
b e c
Ringer Alkaline
1/5 a d
b e c
Ringer Acid
1/5 t

PLATE 24.

Fig. A.—Dorsal view of the skull found in Lenawee county, Michigan. (No. 1634, Smithsonian collection, National Museum, Washington.) The dovetailed recess at the posterior end of the nasals is well shown, on the median line, to receive the blunt wedge-shaped portion of the frontal bone. Note the difference between this part and the nasals of the Kansas specimen.

Fig. B.—Dorsal view of the Boicourt, Linn county, Kansas, skull, showing the heavy scoring of the parietals for ligamentary and muscular attachment. Heavy line work denotes restored portions.

PLATE 24.



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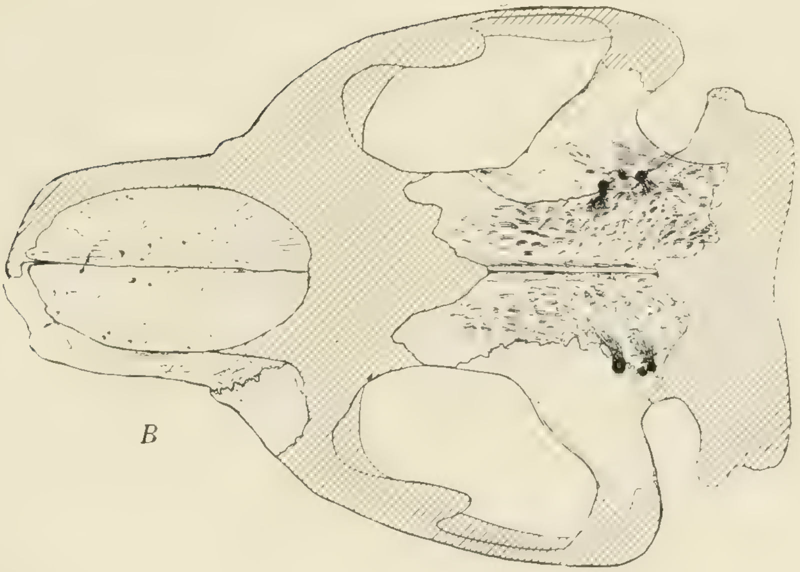


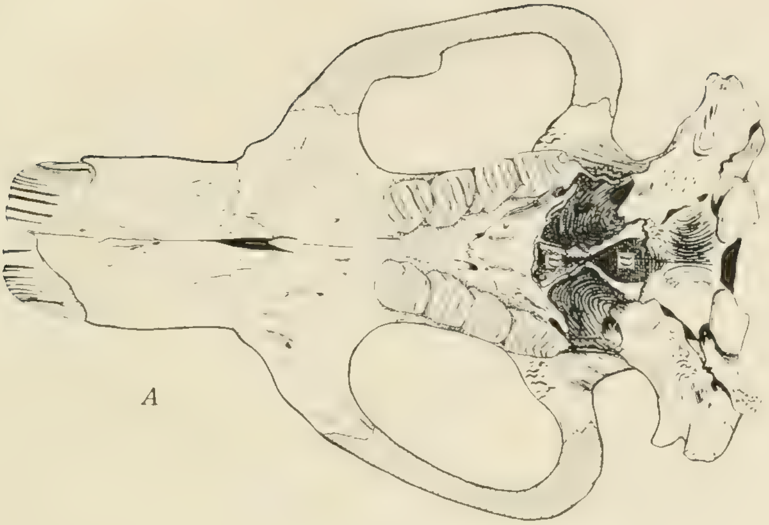
B

PLATE 25.

Fig. A.—Palatine view of the Lenawee skull, showing the double posterior nares, *a-a*, and the thin, laterally compressed pterygoid blades. Line work represents restored portions of the zygomatic arch.

Fig. B.—Palatine view of the Boicourt skull, with incisor removed.

PLATE 25.



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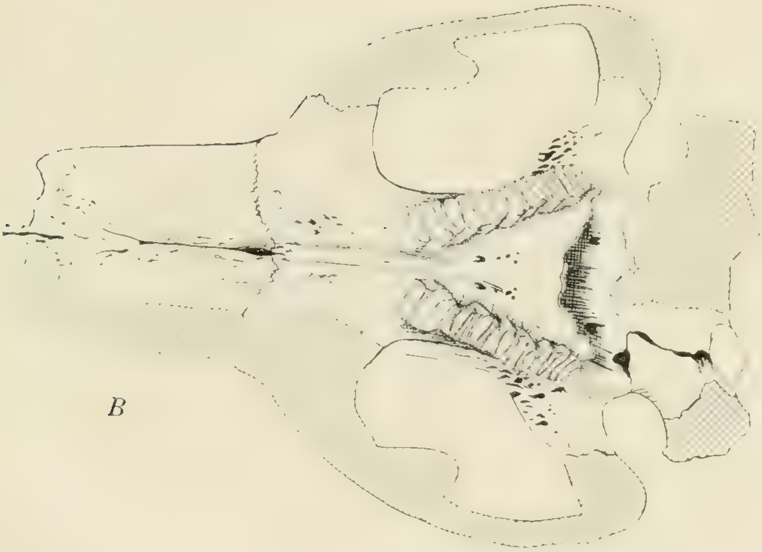


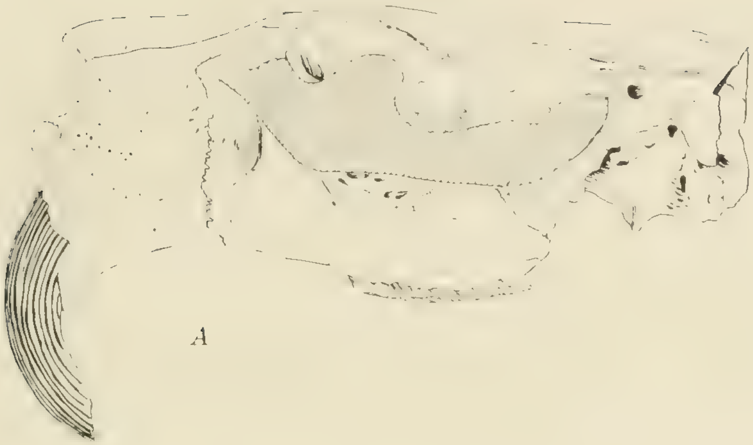
B

PLATES 26, 27.

FIG. A.—Lateral view of the Lenawee skull.

FIG. B.—Photograph of the Boicourt skull, showing the large incisor, *a*, withdrawn from its socket to show the full depth of the cavity; *c* is the frontal sinus, *d* the inner wall removed to show the last molar, *e* the very small brain cavity.

PLATES 26, 27.



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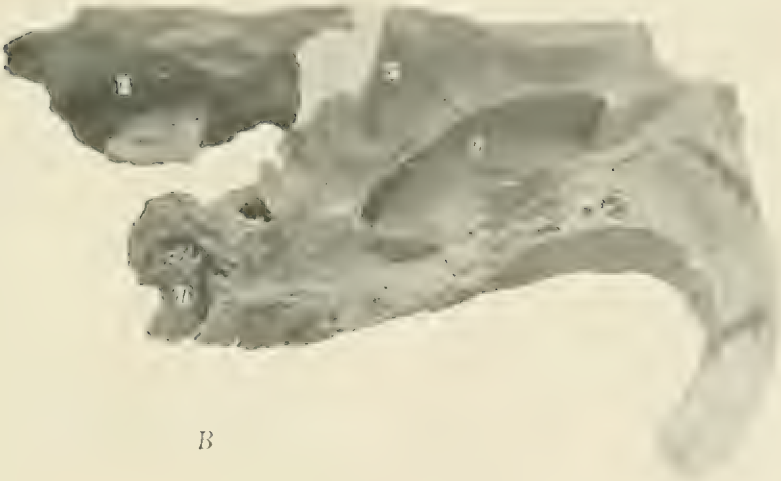


B

PLATE 28.

Melanoplus femur-rubrum.

FIG. 1.—Resting spermatogonium.

FIG. 2.—Spireme stage of spermatogonium, with loops of chromatin arising from homogeneous mass of chromatin at nuclean membrane.

FIG. 3.—Further spireme stage, with tendency to longitudinal split.

FIG. 4.—Vesicular stage in late prophase of spermatogonium. Here the accessory is separated from others. Chromosomes are sometimes vacuolated.

FIG. 5.—Spermatogonium, showing the full number of chromosomes in equatorial plate.

FIG. 6.—Somatic cell from follicular epithelium, showing but twenty-two chromosomes.

FIGS. 7 to 11.—Spermatogonia, showing size and arrangement of chromosomes: Figs. 7 and 8, equatorial plates; 9, metaphase; 10, anaphase; 11, telophase.

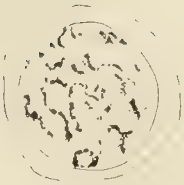
FIGS. 12, 13, 14.—Oögonia in early development: Figure 14, the follicular cells are forming around the young egg; the chromatin is loosely arranged in a ring.

FIGS. 15, 16.—Early resting phase of first spermatocyte.

FIGS. 17, 18.—Spireme stages in first spermatocyte.

FIGS. 19, 20.—Spireme has a distinctly beaded appearance.

PLATE 28.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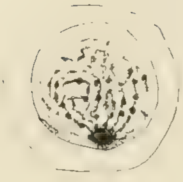
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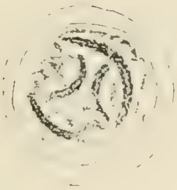
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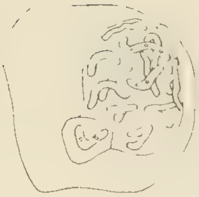
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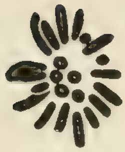
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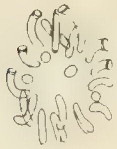
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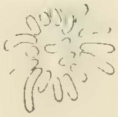
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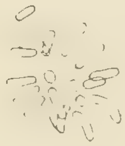
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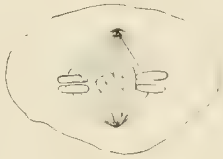
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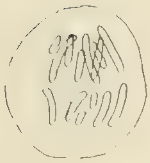
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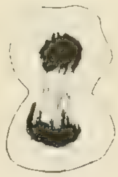
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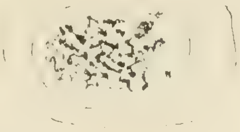
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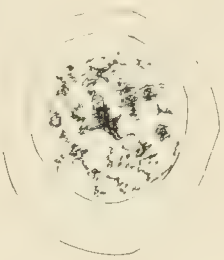
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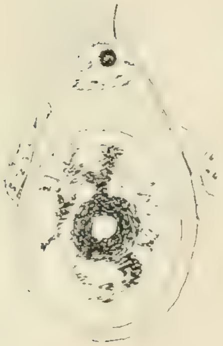
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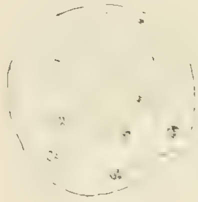
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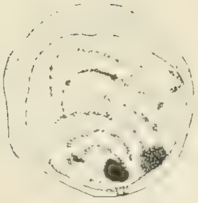
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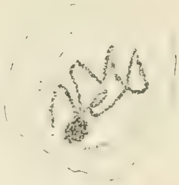
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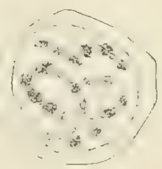
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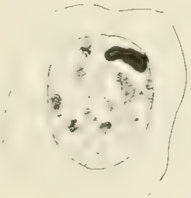
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PLATE 29.

Melanoplus femur-rubrum.

FIG. 1.—First spermatocyte, showing the centrosome in the cytoplasm.

FIG. 2.—Early first spermatocyte, showing the chromatin in large, loose rings.

FIGS. 3 to 9.—Prophases of first spermatocytes, with the ring formations and crosses.

FIG. 9.—The full number of 12 chromosomes are seen here in prophase.

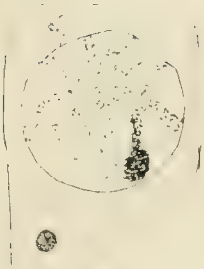
FIGS. 10 to 13.—Polar views of first spermatocyte, showing full number of twelve chromosomes and the tendency toward ring formation.

FIGS. 14 to 18.—Metaphases of firsts.

FIG. 19.—Anaphase of the first division.

FIG. 20.—Early telophase of first spermatocyte.

PLATE 29.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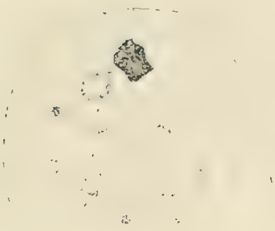
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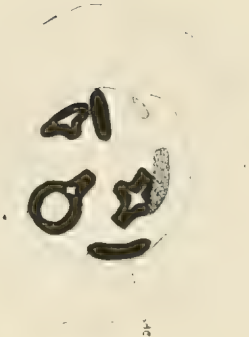
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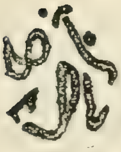
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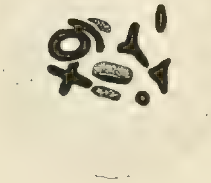
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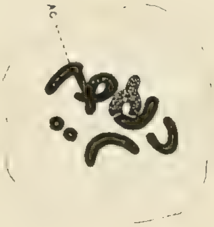
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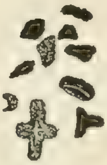
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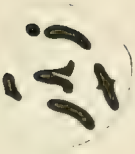
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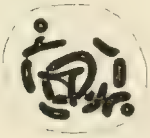
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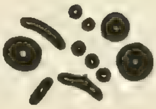
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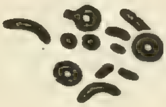
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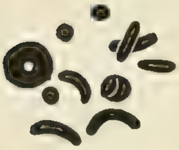
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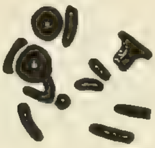
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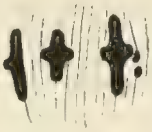
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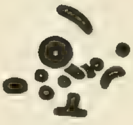
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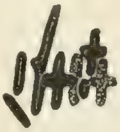
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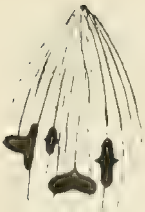
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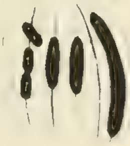
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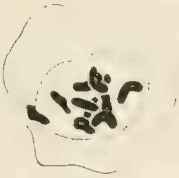
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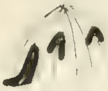
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PLATE 30.

Melanoplus femur-rubrum.

FIGS. 1 to 4.—Metaphase of first division; accessory further toward one pole than the other; arrangement of chromosomes in spindle well shown here; the early ones lying parallel with equatorial plate, and the later ones at right angles.

FIGS. 5, 6.—Metaphases of the second spermatocyte, showing twelve chromosomes.

FIGS. 7, 8.—Metaphases of second showing eleven chromosomes.

FIGS. 9, 10.—Show the early anaphase of second.

FIG. 11.—Late telophase of second spermatocyte.

FIG. 12.—Later stage still of the telophase of the second spermatocyte.

FIG. 13.—Very early spermatid.

FIGS. 14, 15, 17.—Metamorphosis of spermatids.

FIGS. 18, 19, 20.—Dwarfed spermatocytes first and second.

FIGS. 21, 22, 23, 24, 25.—Spermatids in order of development.

PLATE 30.

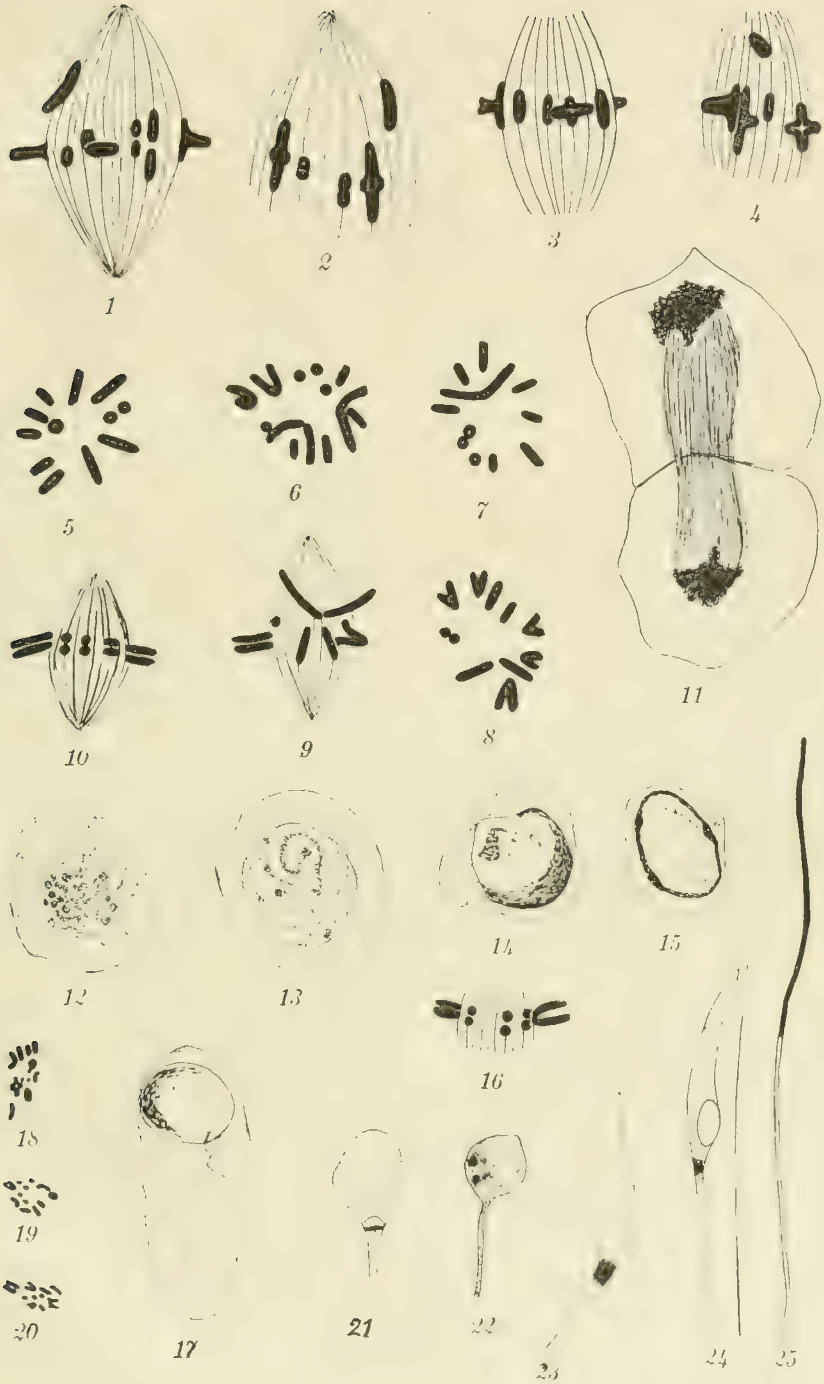
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PLATE 31.

Melanoplus differentialis.

FIGS. 1 to 4.—Spermatogonia, with full number of twenty-three chromosomes seen in equatorial plate.

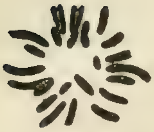
FIGS. 5 to 14.—First spermatocytes seen in equatorial plate; rings vary from one to two.

FIGS. 15, 19, 20, 21, 25, 26.—Side view of chromosomes and spindle.

FIGS. 22, 23.—Equatorial plates, with second spermatocyte from the same mother cell. One shows the eleven chromosomes while the other shows twelve.

FIG. 24.—Equatorial plate of second spermatocytes, showing twelve chromosomes.

PLATE 31.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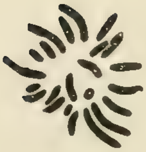
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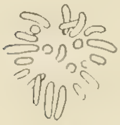
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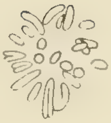
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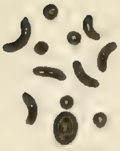
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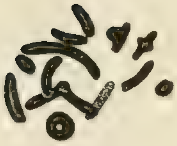
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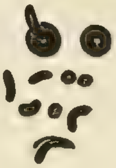
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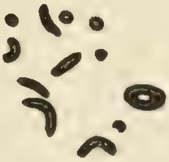
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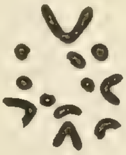
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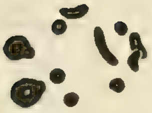
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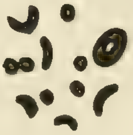
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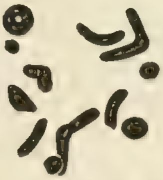
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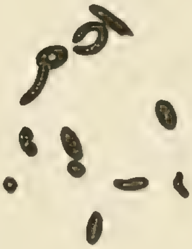
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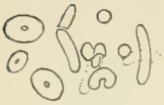
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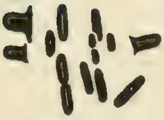
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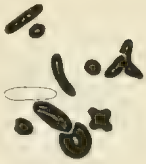
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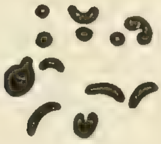
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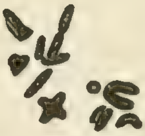
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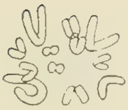
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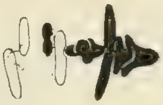
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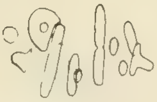
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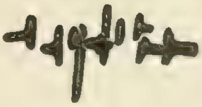
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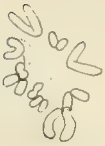
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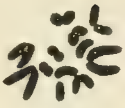
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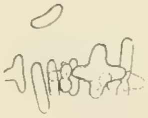
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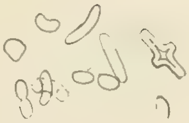
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PLATE 32.

M. atlanis.

FIGS. 1 to 12.—Stages in spermatogenesis of *atlanis:* Fig. 1, spermatocyte in late anaphase; figs. 2, 3, 4, prophases of first division; figs. 5, 6, 7, 8, 9, lateral view of spindle; figs. 10, 11, 12, equatorial view of chromosomes.

M. packardii.

FIGS. 13 to 24.—Stages in spermatogenesis of *packardii.*

FIGS. 13, 14.—Prophase of first spermatocyte.

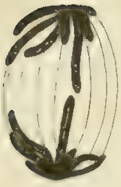
FIGS. 15, 16, 17, 20, 21.—Metaphase of first spermatocyte.

FIGS. 18, 19.—Anaphase of first spermatocyte.

FIGS. 22, 23.—Metaphases of second spermatocyte, with eleven chromosomes.

FIG. 24.—Metaphases of second spermatocyte, with twelve chromosomes.

PLATE 32.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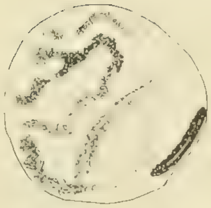
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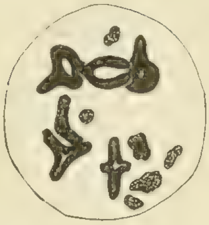
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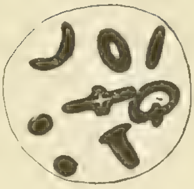
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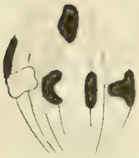
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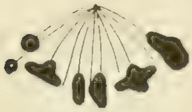
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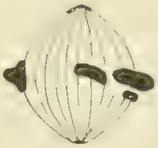
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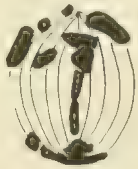
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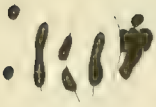
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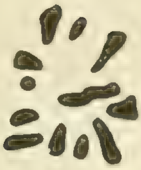
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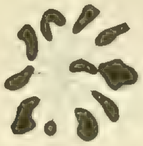
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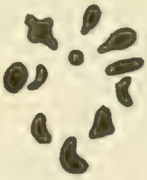
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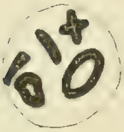
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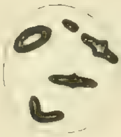
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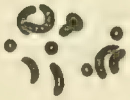
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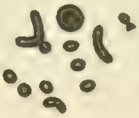
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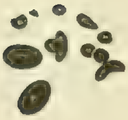
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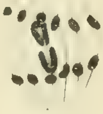
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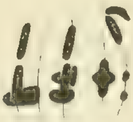
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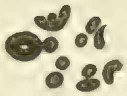
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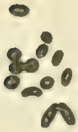
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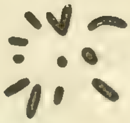
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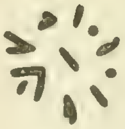
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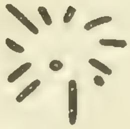
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